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Hilbert space of curved $\beta\gamma$ systems on quadric cones

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Abstract: We clarify the structure of the Hilbert space of curved $\beta\gamma$ systems defined by a quadratic constraint. The constraint is studied using intrinsic and BRST methods, and their partition functions are shown to agree. The quantum BRST cohomology is non-empty only at ghost numbers 0 and 1, and there is a one-to-one mapping between these two sectors. In the intrinsic description, the ghost number 1 operators correspond to the ones that are not globally defined on the constrained surface. Extension of the results to the pure spinor superstring is discussed in a separate work.

Keywords: Conformal Field Models in String Theory, BRST Symmetry.

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Contents

1. Introduction

About seven years ago, a new formalism for the superstring which achieves manifest ten dimensional super-Poincaré covariance was proposed [1]. As of today, the formalism has passed various consistency checks and has been used to compute multiloop amplitudes and to describe Ramond-Ramond backgrounds in a super-Poincaré covariant manner.

One of the key ingredients of the formalism is the use of a bosonic variable λ^α that is constrained non-linearly to be a pure spinor $\lambda\gamma^\mu\lambda = 0$. In a sense, λ^α can be thought as the ghost for the Green-Schwarz-Siegel worldsheet constraint d_α. Although the use of such a constrained ghost system is unconventional, it can be used to construct vertex operators and to define string amplitudes as worldsheet correlation functions [1−3]. Dependence of the amplitudes on the non-zero modes of λ^α and its conjugate ω_α is fixed by the operator product expansions, and the functional integral over the zero-modes can be inferred by requiring BRST and super-Poincaré invariance.

Although the basic ingredients for computing on-shell amplitudes are already there, it would be useful to understand the functional integral over λ^α without relying on the BRST invariance, or equivalently, to understand the nature of the Hilbert space in the operator formalism. This would be necessary, for example, if one wishes to apply the formalism to construct a super string field theory.

There are two basic strategies to study the structure of the Hilbert space for the pure spinors. The first is to deal directly with the constrained variables, and define the Hilbert space as the space of operators that are consistent with the pure spinor constraint [1]. To be consistent with the constraint, the conjugate ω_α has to appear in combinations invariant under the "gauge transformations" $\delta_\Lambda\omega_\alpha = \Lambda^\mu(\gamma_\mu\lambda)_\alpha$ generated by the constraint $\lambda\gamma_\mu\lambda$. The other is to try to remove the constraint by introducing BRST ghosts. The constraint is then expressed effectively as the cohomology condition of the BRST operator D [4].[1]

Each method has its own advantages and disadvantages at the present time. For the first method, the theory of so-called curved $\beta\gamma$ systems provide a natural framework to deal with the constraint [6−9]. The basic idea is to regard the pure spinor sector as a collection of free bosonic $\beta\gamma$ systems defined locally but intrinsically on the pure spinor target space. Although this Čech type formulation provides a nice description of the pure spinor sector, self-contained rules for performing the functional integral over the fields defined only locally remains to be clarified. The BRST method for the pure spinor system, on the other hand, meets more severe difficulties. Since the pure spinor constraint is infinitely reducible (meaning there are relations among the constraints, and relations-for-relations and so on)

[1] D should not to be confused with the "physical" BRST operator $Q = \int \lambda^\alpha d_\alpha$ of the pure spinor formalism. (Because a possible use of D is to combine it with Q to construct a single nilpotent operator $\hat{Q} = D + Q + \cdots$, we called D a "mini-BRST" operator in [5].)

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one has to introduce an infinite chain of ghosts-for-ghosts [4]. Although the infinite ghosts in fact are fairly useful for computing partition functions [10, 5], expressions for the vertex operators and the composite reparameterization b-ghost become complicated and at best rather formal.

Taking aim at clarifying the Hilbert space for the pure spinors, we in this paper consider models with a single irreducible constraint $\lambda^i\lambda^i = 0$ ($\lambda^i \not\equiv 0$, $i = 1 \sim N$). It will be argued that the curved $\beta\gamma$ and BRST formalisms provide equivalent classical descriptions of the system, although, quantum mechanically, the Hilbert spaces of the two descriptions differ slightly due to the different normal ordering prescriptions used. Nevertheless, since our partition function (in fact an index $\mathrm{Tr}[(-1)^F \cdots]$) is defined so that it is insensitive to quantum corrections, the two descriptions lead to the same partition function even quantum mechanically. We shall use the partition function as a guide to study the structure of the Hilbert space.

The BRST formalism is designed so that the ghost number 0 cohomology of the BRST operator $D = \int b(\lambda\lambda)$ reproduces the usual gauge invariant operators, that is, the gauge invariant polynomials made out of λ^i and its conjugate ω_i, and their derivatives. In the curved $\beta\gamma$ language, those gauge invariant polynomials are nothing but the globally defined operators[2], so one expects the agreement on the ghost number 0 sector as has been noted in [13]. In this paper, we will claim that the equivalence goes beyond the ghost number 0 sector. For example, the BRST ghost itself b (ghost number $+1$) is clearly in the cohomology of $D = \int b(\lambda\lambda)$. In the curved $\beta\gamma$ description, b will be identified as an operator that is defined only on single overlaps of the coordinate charts, or in other words, as an element of the first Čech cohomology.

The fact that the number of coordinate overlap corresponds to the BRST ghost number can be best understood in the so-called non-minimal or Dolbeault formulation of the curved $\beta\gamma$ systems. In this formulation, one introduces the complex conjugate $\overline{\lambda}_i$ of λ^i and its differential $r_i = \mathrm{d}\overline{\lambda}_i$, together with their conjugates $\overline{\omega}^i$ and s^i. The relevant cohomology operator is an extension of the Dolbeault differential $\overline{\partial}_X = -r_i\overline{\omega}^i \sim \mathrm{d}\overline{\lambda}_i(\partial/\partial\overline{\lambda}_i)$. Then, an object defined only on the nth overlaps (n-cochain) will be identified as an n-form defined on the total space. Note that this identification is consistent with the expected statistics. For example, the fermionic ghost b is identified as a 1-form which is anticommuting.

The way we relate the (classical) BRST and Dolbeault/Čech cohomologies is as follows. First, we embed both the BRST and Dolbeault cohomologies to that of the combined operator $D + \overline{\partial}_X$. Then, BRST and Dolbeault cohomologies are nothing but the special gauge choices in the $D + \overline{\partial}_X$ cohomology, where non-minimal variables are absent (BRST), and BRST ghosts are absent (Dolbeault). Going back and forth between Čech and Dolbeault languages can be achieved by imitating the standard argument in complex analysis, i.e. by using a partition of unity to patch together Čech cochains to obtain Dolbeault forms. Although we will not explore in the main text, it should be possible to directly relate the (minimal) BRST and Čech languages by considering the cohomology of $D + \check{\delta}$, where $\check{\delta}$ is

[2] In lower dimensions $N \leq 3$, there are globally defined operators which cannot be described as gauge invariant polynomials [12].

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the difference operator of Čech cohomology.

One of the virtue of studying these simpler models is that the BRST description is very effective, allowing a close study of its cohomology. In particular, the full partition function of the BRST cohomology can be easily computed and it manifestly possesses two important symmetries that we shall call "field-antifield" and "$*$-conjugation" symmetries. The former implies that, after coupling to "matter" variables (p_i, θ^i), the cohomology of the "physical" BRST operator $Q = \int \lambda^i p_i$ comes in field-antifield pairs.[3] As such, the symmetry is indispensable when one tries to define a sensible "spacetime" amplitudes.

The second symmetry, the $*$-conjugation symmetry, turns out to be more powerful for analyzing the structure of the BRST cohomology. It implies the existence of a non-degenerate inner product that couples the cohomologies at ghost numbers k and $1-k$. In particular, there is a one-to-one mapping between $H^0(D)$ and $H^1(D)$, and since $H^k(D)$ is empty for k negative, all the higher cohomologies $H^k(D)$ with $k > 1$ are also empty. This "vanishing theorem" is rather important for the pure spinor case (H^k with $k > 3$) [5].

The plan of this paper is as follows. In section 2, after briefly reviewing the general theory of the curved $\beta\gamma$ formalism, we introduce the models to be considered in this paper, both in curved $\beta\gamma$ and BRST descriptions. As mentioned above, they are modeled after the ghost sector of the pure spinor superstring, and the target spaces are simple cones defined by a single quadratic constraint.

In section 3 we compare the partition functions of naive gauge invariant polynomials and that of the BRST cohomology, and find that the latter includes some extra states. In fact, those "extra" states are essential for having field-antifield symmetry, so perhaps it is more appropriate to refer to them as the states "missing" from the space of naive gauge invariant polynomials.

In section 4, we study in detail the structure of the quantum BRST cohomology. It will be found that there is a one-to-one mapping between the gauge invariant polynomials (elements of $H^0(D)$) and the "extra" states (elements of $H^1(D)$). Also, it will be shown that the quantum BRST cohomology is empty outside those degrees.

Finally, in section 5, the mapping between BRST and Čech/Dolbeault curved $\beta\gamma$ descriptions is explained. We shall show explicitly how the classical pieces of the cohomology representatives are related and point out how this correspondence can be broken quantum mechanically.

An appendix is provided for explaining some details of the curved $\beta\gamma$ description of the models considered in this paper.

2. The models

We begin with a brief review of the basics of the theory of curved $\beta\gamma$ systems, following [6–9]. The formalism is then used to introduce the models by specializing the target space to be a simple quadric cone $\lambda^i\lambda^i = 0$ ($i = 1 \sim N$). The BRST descriptions of the same models

[3] This fact and some topics related to our paper have been recently reported in [4] for the simple model $N = 2$.

are introduced in section 2.3, and the geometries of the target space for some specific values of N are explained in section 2.4.

2.1 Quick review of the curved $\beta\gamma$ formalism

To construct a general curved $\beta\gamma$ system on a complex manifold X, one usually starts with a set of free conformal field theories taking values in the coordinate patches $\{U_A\}$ of X, and tries to glue them together. The field contents of each conformal field theory are the (holomorphic) coordinate of a patch u^a and its conjugate v_a satisfying the free field operator product expansion

$$u^a(z)v_b(w) = \frac{\delta^a{}_b}{z-w}\,. \tag{2.1}$$

Unlike the conventional sigma models on complex manifolds, antiholomorphic coordinates need not be introduced. On an overlap $U_A \cap U_B$, two coordinates u^a and $\tilde{u}^{\tilde{a}}$ are related in the usual geometric manner,

$$\tilde{u}^{\tilde{a}} = \tilde{u}^{\tilde{a}}(u)\,, \tag{2.2}$$

but it requires some thought to find the gluing condition for the conjugates v_a and $\tilde{v}_{\tilde{a}}$ because the classical relation,

$$\tilde{v}_{\tilde{a}} \overset{?}{=} \tau_{\tilde{a}}{}^b v_b\,, \quad \left(\tau_{\tilde{a}}{}^b = (\tau_{AB})_{\tilde{a}}{}^b = \frac{\partial u^b}{\partial \tilde{u}^{\tilde{a}}}\right), \tag{2.3}$$

suffers from normal ordering ambiguities. In order to glue the free field operator products (2.1) on an overlap, the conjugates in two patches must be related as [6 – 9]

$$\tilde{v}_{\tilde{a}} = {:}\tau_{\tilde{a}}{}^b v_b{:} + \tilde{\phi}_{\tilde{a}\tilde{b}}\partial_z \tilde{u}^{\tilde{b}}\,, \tag{2.4}$$

where the correction $\tilde{\phi}$ is a matrix defined on the overlap and ${:}\tau_{\tilde{a}}{}^b v_b{:} = {:}(\partial u^b/\partial \tilde{u}^{\tilde{a}})v_b{:}$ denotes the usual free field normal ordering with respect to u and v. (There are no ordering ambiguities for $\tilde{\phi}_{ab}\partial_z \tilde{u}^b$.) It is convenient to decompose $\tilde{\phi}$ into symmetric and antisymmetric pieces,

$$\tilde{\phi}_{\tilde{a}\tilde{b}} = \tilde{\sigma}_{\tilde{a}\tilde{b}} + \tilde{\mu}_{\tilde{a}\tilde{b}}\,, \tag{2.5}$$

and regard the antisymmetric piece $\tilde{\mu}_{\tilde{a}\tilde{b}}$ as the component of a two form

$$\mu = \frac{1}{2}\tilde{\mu}_{ab}\mathrm{d}\tilde{u}^a \wedge \mathrm{d}\tilde{u}^b\,. \tag{2.6}$$

Solving $\tilde{v}_{\tilde{a}}(z)\tilde{v}_{\tilde{b}}(w) = 0$, one finds the conditions on $\tilde{\sigma}$ and μ to be

$$\begin{aligned}
\tilde{\sigma}_{\tilde{a}\tilde{b}} &= -(\partial_c \tau_{\tilde{a}}{}^d \partial_d \tau_{\tilde{b}}{}^c) = -\left(\frac{\partial^2 u^d}{\partial u^c \partial \tilde{u}^{\tilde{a}}}\frac{\partial^2 u^c}{\partial u^d \partial \tilde{u}^{\tilde{b}}}\right), \\
\mathrm{d}\mu &= -\mathrm{tr}(\tau^{-1}\mathrm{d}\tau)^3 = -\frac{\partial \tilde{u}^{\tilde{a}}}{\partial u^b}\mathrm{d}\left(\frac{\partial u^b}{\partial \tilde{u}^{\tilde{c}}}\right) \wedge \frac{\partial \tilde{u}^{\tilde{c}}}{\partial u^d}\mathrm{d}\left(\frac{\partial u^d}{\partial \tilde{u}^{\tilde{e}}}\right) \wedge \frac{\partial \tilde{u}^{\tilde{e}}}{\partial u^f}\mathrm{d}\left(\frac{\partial u^f}{\partial \tilde{u}^{\tilde{g}}}\right).
\end{aligned} \tag{2.7}$$

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The argument up to this point was local and the quantum correction $\phi\partial u$ can always be found. The 2-form μ is the data assigned to every double overlaps $U_{AB} = U_A \cap U_B$ so it constitutes a Čech 1-cochain; when we wish to emphasize this fact, we denote $\mu = (\mu_{AB})$ etc. Now, the solution to the gluing condition (2.7) is not quite unique and, at the same time, might not be compatible on the triple overlaps $U_{ABC} = U_A \cap U_B \cap U_C$. The ambiguity comes from the freedom to add closed 2-form valued Čech 1-coboundaries to μ

$$\mu = (\mu_{AB}) \to \mu + \check{\delta}\alpha = (\mu_{AB} + \alpha_A - \alpha_B)\,, \quad \alpha = (\alpha_A)\colon \text{closed 2-form}\,, \tag{2.8}$$

which can be absorbed in the redefinitions of the local coordinates (and their conjugates) in U_A and U_B. On the other hand, the consistent gluing requires the following 2-cocycle be a coboundary:

$$\psi = (\psi_{ABC}) = \big(\mu_{AB} + \mu_{BC} + \mu_{CA} - \mathrm{tr}(\tau_{AB}\, \mathrm{d}\tau_{BC} \wedge \mathrm{d}\tau_{CA})\big)\,. \tag{2.9}$$

In short, the moduli of the gluing is parameterized by the first Čech cohomology $H^1(\mathcal{Z}^2)$ of closed 2-forms on X, but it can be obstructed by the second cohomology $H^2(\mathcal{Z}^2)$ (or the first Pontryagin class $p_1(X)$). Also, the gluing of the global symmetry currents of X are parameterized and possibly obstructed by similar ("equivariant version" of) cohomologies.

Finally, let us recall that even if the operator products (2.1) and the symmetry currents could be consistently glued, one may not be able to define the energy-momentum tensor T globally. This implies the violation of the conformal symmetry due to an anomaly. For T to be well-defined, one has to improve it using a nowhere vanishing holomorphic top form Ω of X. The obstruction to having Ω is the first Chern class $c_1(X)$. Hence to have a globally defined conformal field theory as a curved $\beta\gamma$ system, the target X must be a Calabi-Yau space (though strictly speaking X need not be Kähler).

This concludes our brief review of the basic notions of the theory of curved $\beta\gamma$ systems.

2.2 Curved $\beta\gamma$ description

From now on, we specialize to a subset of curved $\beta\gamma$ systems where the target space X is a cone in $\mathbb{C}^N$ defined by a quadratic constraint [13]

$$X = \{\lambda^i \in \mathbb{C}^N \mid G \equiv \lambda^i \gamma_{ij} \lambda^j = 0\,, \quad \lambda \neq 0\}\,, \quad (i\,, j = 1 \sim N)\,. \tag{2.10}$$

Here, γ_{ij} is some non-degenerate symmetric constant "metric". Of course, one can always diagonalize as $\gamma_{ij} = \delta_{ij}$ so we drop the factor of γ and its inverse, and do not distinguish upper and lower indices.

Since we remove the origin $\lambda = 0$ as indicated above, X is a $\mathbb{C}^*$-bundle over the base $B = X/\mathbb{C}^*$ where the quotient acts by the global rescaling of λ. The target space reparameterization (Pontryagin) anomaly is absent just as in the pure spinor case. That is, although the base B has a non-trivial anomaly 2-cocycle ψ (2.9), its extension to the total space X represents a trivial Čech class by virtue of the fiber direction (see appendix A) [9]. Therefore, the conjugate ω_i, or more precisely its independent components, can be glued consistently. For the case at hand, the symmetry currents for the SO(N) rotation N_{ij}

and rescaling of the cone J can also be defined consistently.[4] Furthermore, X is a (non-compact) Calabi-Yau space admitting a nowhere vanishing holomorphic top form. Thus, the energy-momentum tensor can be globally defined and the curved $\beta\gamma$ theory on X is conformally invariant.

For completeness, we give in appendix A some more details of the curved $\beta\gamma$ description such as the choice of local coordinates and the expressions of the currents (J, N, T) etc.

Non-minimal or Dolbeault description When dealing with operators that are not globally defined, it is notationally more convenient to introduce the non-minimal variables defined as [15]

$$\begin{aligned} \overline{\lambda}_i\,,\ \overline{\omega}^i\,, \quad &\big(\overline{\lambda}_i\overline{\lambda}_i = 0\,, \quad \delta_{\Lambda,\Psi}\overline{\omega}^i = \Lambda\overline{\lambda}^i + \Psi r^i\big)\,, \\ r_i = \mathrm{d}\overline{\lambda}_i\,,\ s^i\,, \quad &\big(r_i\overline{\lambda}_i = 0\,, \quad \delta_\Lambda s^i = \Lambda\overline{\lambda}^i\big)\,. \end{aligned} \tag{2.11}$$

Observe that $\overline{\lambda}$ satisfies the same constraint as λ. In the language of complex geometry, $\overline{\lambda}$ corresponds to the antiholomorphic coordinate of the target space X. The virtue of introducing those extra variables is that one can deal with globally defined operators, often hiding the explicit dependence on the local coordinates. The mapping between Čech and Dolbeault descriptions can be explicitly done using the partition of unity given in appendix A.

Physical states in non-minimal formalism are defined as the cohomology of the Dolbeault operator

$$\overline{\partial}_X = -r_i\overline{\omega}^i \sim \mathrm{d}\overline{\lambda}_i\frac{\partial}{\partial\overline{\lambda}_i}\,. \tag{2.12}$$

If one wishes to be rigorous, this gauge invariant expression should be understood in terms of the local coordinates. Despite its simple form, the cohomology of $\overline{\partial}_X$ is not quite trivial, because the minimal variables are constrained, and because one allows poles in $(\lambda\overline{\lambda})$.

However, non-zero modes of the non-minimal variables do not affect the cohomology due to the relation

$$\overline{\partial}_X(s\partial\overline{\lambda}) = \overline{\omega}\partial\overline{\lambda} + s\partial r = -T_{\text{non-min}}\,. \tag{2.13}$$

Whenever there is a $\overline{\partial}_X$-closed operator F with positive weight h carried by the non-minimal sector, it is also a $\overline{\partial}_X$ of itself multiplied by the zero-mode of $s\partial\overline{\lambda}$:

$$-\frac{1}{h}\overline{\partial}_X\big((s\partial\overline{\lambda})_0F\big) = F\,. \tag{2.14}$$

Similarly, due to the relation

$$\overline{\partial}_X(s\overline{\lambda}) = \overline{\omega}\overline{\lambda} + sr = -J_{\text{non-min}}\,, \tag{2.15}$$

[4] J is often called as "ghost number" current in the literature. But we shall call it "t-charge current" instead to avoid the confusion with the BRST ghost number introduced later.

the zero-modes of $\overline{\lambda}$ and r can only appear in the non-minimal charge 0 combinations

$$(\lambda\overline{\lambda})^{-1}\overline{\lambda}_i \quad \text{and} \quad (\lambda\overline{\lambda})^{-1}r_i\,. \tag{2.16}$$

Given those restrictions on the appearance of non-minimal variables, it follows that whether they are constrained or not is irrelevant for the cohomology of $\overline{\partial}_X$. That is, even if one regards the non-minimal variables as *unconstrained*, the cohomology of $\overline{\partial}_X$ remains unchanged.

"Gauge invariance" in curved $\beta\gamma$ framework. When discussing the constrained curved $\beta\gamma$ systems above, we used the notion of "gauge invariance" to define the space on which the Čech or Dolbeault operators act. In the curved $\beta\gamma$ framework, however, one usually deals with the operators defined intrinsically on the target space X (even in the non-minimal language), and does not worry about the "gauge invariance". Let us explain the relation between the two descriptions.

For simplicity, consider the particle moving on the cone X. When one speaks of the gauge transformation $\delta_\Lambda\omega_i = \Lambda\lambda_i$, it is implicitly assumed that the phase space T^*X is embedded in a Euclidean space $(\omega, \lambda) \in T^*\mathbb{C}^N = \mathbb{C}^{2N}$. Then, a gauge transformation generates a motion vertical to T^*X, and the gauge invariance of an object simply means that it is living inside T^*X. In the curved $\beta\gamma$ language, T^*X is treated intrinsically and everything is manifestly gauge invariant; there is really no way to construct "gauge *non*-invariant object" just by using the local coordinates on T^*X. Therefore, "gauge invariance" is a convenient way to refer to the operators defined intrinsically on X, but by using the "extrinsic" coordinates (ω, λ).

Note, however, that the converse is not necessarily true. For example, there can be operators that are globally defined on X, but nevertheless cannot be described as a gauge invariant polynomial in $(\omega, \lambda; \partial)$. For the class of models considered in the present paper, this will happen when the dimension of X is smaller than 3 ($N < 4$), i.e. when the base B of X has one or "zero" dimensions.

The reason why we find it useful to work in the space of "gauge invariant" functions of $(\omega, \lambda; \partial)$ is the following. Later in section 5 we explore the relation between the BRST and the intrinsic curved $\beta\gamma$ descriptions of the constraints. Since (ω, λ) are promoted to genuine free fields in the BRST framework, what naturally appears there is X embedded in a flat space, rather than its intrinsic description.

2.3 BRST description

For the model with the irreducible constraint (2.10) the conventional BRST formalism provides a very simple way of describing it, compared to the elaborate language of the curved $\beta\gamma$ formulation. (This is not necessarily the case for infinitely reducible constraints such as the ones for the pure spinors.) Here, a fermionic (b, c) ghost pair is introduced to impose the constraint effectively and the physical states are described as the cohomology of the BRST operator

$$D = \int b(\lambda\lambda)\,. \tag{2.17}$$

The ghost number 0 cohomology $H^0(D)$ reproduces the space of globally defined gauge invariant polynomials. However, there are also non-trivial cohomologies at non-zero ghost numbers. Typical example is the ghost b itself with ghost number $+1$. Describing explicitly the operator corresponding to b in the curved $\beta\gamma$ language is one of the goals of the present paper.

As will be shown in section 4, the cohomology turns out to be non-vanishing only at ghost numbers 0 and 1. Moreover, we find that the elements of $H^0(D)$ and $H^1(D)$ are paired under a certain inner product.

We expect that this property of the BRST cohomology is a general property of the theory defined by a system of homogeneous constraints. In particular, for the important case of the pure spinor model the non-vanishing cohomologies are $H^0(D)$ and $H^3(D)$, which again come in pairs [5].

2.4 Geometries of X and models with lower values of N

In the forthcoming sections we will assume $N \geq 4$ and our discussions will not depend on the specific value of N. One can define consistent models for $2 \leq N \leq 3$ both in the BRST and the curved $\beta\gamma$ frameworks, but the two descriptions will not be equivalent even classically (at least when they are defined analogously to the $N \geq 4$ case). Here, we explain the geometry of X for some values of N, and give a rationale behind the restriction on N. Appendix A provides some additional properties of X such as coordinate systems and the associated partition of unity etc.

$\boldsymbol{\lambda^2 = 0}$ $\boldsymbol{(N = 1)}$. The model with $N = 1$ has a single coordinate variable λ which is constrained as $\lambda^2 = 0$. As such, the "target space" is not geometrical in the usual sense, and it is not clear what local coordinates one should take to define the curved $\beta\gamma$ model intrinsically.

Also on the BRST side, this model is qualitatively different from $N \geq 2$ models because λ^2 and its derivatives $\partial^n\lambda^2$ are not independent. Therefore, the naive BRST charge $D = \int b\lambda^2$ has extra cohomologies outside $H^0(D)$ and $H^1(D)$. By appropriately introducing a chain of ghosts-for-ghosts, one should be able to describe the gauge invariant polynomials in (ω, λ) as the zeroth cohomology. But let us avoid this effort in the present paper, since we explain the BRST construction for the reducible pure spinor constraints in detail in [5]. Instead, we explicitly identify some unwanted cohomology elements in section 3.5.

$\boldsymbol{\lambda\tilde{\lambda} = 0}$ $\boldsymbol{(N = 2)}$. For $N \geq 2$, the constraint $\lambda^i\lambda^i = 0$ is irreducible so the BRST operator should be given by $D = \int b(\lambda\lambda)$ and the structure of its cohomology does not depend on N. Also, the space X defined by the constraint is non-degenerate and the curved $\beta\gamma$ system on X is consistent. However, for $N = 2, 3$ models the two descriptions do not agree because the intrinsic curved $\beta\gamma$ description allows some (globally defined) operators which cannot be described as the polynomials of the extrinsic coordinates (ω, λ).

Defining

$$(\lambda, \tilde{\lambda}) = (\lambda^1 + i\lambda^2, \lambda^1 - i\lambda^2)\,, \tag{2.18}$$

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the constraint for the $N = 2$ model can be expressed as $\lambda\tilde{\lambda} = 0$. So the geometry of X is a simple cone, but it becomes a union of two disjoint components when the origin is removed. As such, the intrinsic description of the system on X is quite different from its embedding in the flat space (ω, λ), and hence from the BRST description. (The BRST treatment of this model and the enumeration of gauge invariant polynomials up to level 2 was studied in [13].)

$xy - z^2 = 0$ ($N = 3$). Similarly, the constraint for the $N = 3$ model can be rephrased as $xy - z^2 = 0$ where the new variables are defined as

$$(x, y, z) = (\lambda^1 + i\lambda^2, \lambda^1 - i\lambda^2, i\lambda^3)\,. \tag{2.19}$$

The space X is in fact a simple singular Calabi-Yau space

$$\mathbb{C}^2/\mathbb{Z}^2\,, \tag{2.20}$$

which has a so-called A_1 singularity at the origin $(x, y, z) = (0, 0, 0)$. This can be seen by using the coordinate $(a, b) \in \mathbb{C}^2$ and defining

$$(x, y, z) = (a^2, b^2, ab)\,. \tag{2.21}$$

The division by $\mathbb{Z}^2$ identifies a point (a, b) with the antipodal point $(-a, -b)$.

Although the curved $\beta\gamma$ system on X by itself is perfectly sensible, it is not equivalent even classically to the BRST system with $D = \int b(xy - z^2)$. The reason is because, at a given mass level, there is a finite number of globally defined operators on X that cannot be written as gauge invariant polynomials in (x, y, z). For example, as noted in [12], there is one such operator at the first mass level. In the coordinate system $(g, u) \in U_1$ and $(\tilde{g}, \tilde{u}) \in U_{\tilde{1}}$ $(X = U_1 \cup U_{\tilde{1}})$ given in appendix A, the extra state is given by the Čech 0-cocycle

$$F = (F_1, F_{\tilde{1}}) = (g\partial u, -\tilde{g}\partial\tilde{u})\,, \qquad \big(g = \tilde{g}\tilde{u}^2\,,\ u = 1/\tilde{u}\big)\,. \tag{2.22}$$

The coordinate patches U_1 and $U_{\tilde{1}}$ correspond to the region $x \neq 0$ and $y \neq 0$ respectively, and F can also be expressed as [12]

$$F = (zx^{-1}\partial x - \partial z, -zy^{-1}\partial y + \partial z)\,. \tag{2.23}$$

Clearly, there is no corresponding operator in the BRST cohomology computed in the polynomial regime, so the $N = 3$ curved $\beta\gamma$ model is different from the BRST model.

One might worry if there exist non-trivial elements of the Čech cohomology for $N \geq 4$ models $xy - z_a z_a = 0$ $(a = 3 \sim N)$ as well, but it can be argued that there are none. Note that the existence of F crucially depends on the fact that the base B of X is one dimensional. In higher dimensions $(N \geq 4)$, the angular coordinate $u_a \in B$ carries an index and transforms like $u_a = \tilde{u}_a(\tilde{u} \cdot \tilde{u})^{-1}$. So $\partial u_a \in U_1$ have a pole $(\tilde{u} \cdot \tilde{u})^{-2}$ in another patch $U_{\tilde{1}}$, and the only way to cancel the pole is to multiply it by $g^2 = \tilde{g}^2(\tilde{u} \cdot \tilde{u})^2$. But $g^2\partial u_a$ (unlike $g\partial u_a$) is in fact a polynomial $z_a\partial x - x\partial z_a$. Similarly, there should be no non-polynomial operators at higher mass levels.

Another way to understand this is to note that F in (2.23) does not have a corresponding operator on a slightly deformed space $X_\epsilon : xy - z^2 = \epsilon$. That is, the order ϵ term of the deformed operator $F_\epsilon = F + \epsilon F' + \cdots$ has a pole in z and hence is not globally defined on X_ϵ. Therefore, for $N \geq 4$ where the additional coordinates λ^i ($i = 4 \sim N$) can play the role of ϵ, there will not be the extra operators analogous to F.

For those reasons, we assert for $N \geq 4$ models that all the elements of the Čech cohomology can be represented using the extrinsic coordinates (ω, λ), though we do not have a rigorous proof.

$xy - zw = 0$ ($N = 4$). The target space of the $N = 4$ model is the famous conifold as can be seen from defining

$$(x, y, z, w) = (\lambda^1 + i\lambda^2, \lambda^1 - i\lambda^2, i\lambda^3 - \lambda^4, i\lambda^3 + \lambda^4)\,. \tag{2.24}$$

(A partial enumeration of gauge invariant polynomials up to level 2 for this model was studied in [12].)

As explained above, all the models with $N \geq 4$ should behave qualitatively the same. In particular, we shall argue that its curved $\beta\gamma$ description is classically equivalent to the BRST description.

$D = 8$ pure spinor ($N = 8$). We have been implicitly assuming that λ^i transforms as a vector of SO(N). For the special value of $N = 8$, however, λ^i is not significantly different from the SO(8) (chiral) spinor λ^a due to the triality. λ^a satisfying $\lambda^a\lambda^a = 0$ is in fact nothing but the Cartan pure spinor in eight dimensions.

3. Partition function, its symmetries and the extra states

As mentioned in the introduction, the main motivation for the present investigation is to understand the proper Hilbert space for the pure spinor superstring in a simplified setup. We begin the study by computing the partition function of the gauge invariant polynomials, by explicitly counting them at several lower mass levels. Our main finding will be that, starting from the first mass level, the space of naive gauge invariants lacks the field-antifield symmetry because of some finite number *fermionic* operators that are missing.

On the contrary, the partition function of the BRST cohomology is found to enjoy the field-antifield symmetry. Since the ghost number 0 sector of the BRST cohomology is (classically) equivalent to the space of gauge invariant polynomials, this means that the states depending essentially on the ghosts are very important. Those extra states are explicitly identified in section 3.4.

Also, the BRST partition function is found to possess another discrete symmetry which we call "$*$-conjugation symmetry". Both field-antifield and $*$-conjugation symmetries reflect certain dualities of the cohomology, and their existence plays an important role for the consistency of the pure spinor formalism.

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3.1 Definition of the partition function

We begin by describing the definition of our partition function. The characters of the states we are interested in are

- statistics (Grassmanity) measured by $(-1)^F$ (F: fermion number operator),
- weight (Virasoro level) measured by L_0, and
- t-charge measured by a $\mathrm{U}(1)$ charge J_0.

By introducing formal variables (q,t) to keep track of the charges, the partition function is defined as

$$Z(q,t) = \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{L_0} t^{J_0} \,. \tag{3.1}$$

What we are really interested in is the Hilbert space $\mathcal{H}$ in which the trace is taken over, and we shall define the currents for L_0 and J_0 in the next paragraph.

In the BRST framework, basic fields obey free field operator products, and the ghost extended energy-momentum tensor and the t-charge current are defined as

$$T = -\omega_i \partial\lambda^i - b\partial c \,, \quad J = -\omega_i \lambda^i - 2bc \,. \tag{3.2}$$

The charges of the basic operators are

$$F(\omega,\lambda) = (0,0) \,, \qquad h(\omega,\lambda) = (1,0) \,, \qquad t(\omega,\lambda) = (-1,1) \,, \tag{3.3}$$

$$F(b,c) = (1,1) \,, \qquad h(b,c) = (1,0) \,, \qquad t(b,c) = (-2,2) \,. \tag{3.4}$$

In particular, the BRST operator $D = \int b(\lambda\lambda)$ is neutral both under L_0 and J_0, so the partition function of D-cohomology is insensitive to quantum corrections. (Similar remark applies for the Čech/Dolbeault cohomologies for the intrinsic description.)

Let us remark in passing that we define the ghost number current as

$$J_g = +bc \,, \tag{3.5}$$

so that the ghost numbers are

$$g(b,c;D) = (1,-1;1) \,. \tag{3.6}$$

In the curved $\beta\gamma$ framework, construction of T and J are more complicated but their existence is assured by the general theory as we briefly recalled above [6–9]. Their explicit expressions are given in appendix A for completeness. Here, let us only mention that they can be constructed and that the t-charges of operators can be correctly inferred by expressing them in terms of the "extrinsic coordinates" (ω,λ) carrying t-charges $(-1,1)$. For example, the t-charge of $J = -\omega\lambda +$ (quantum corrections) itself is 0.

3.2 Gauge invariant polynomials

We now count the number of gauge invariant polynomials constructed out of λ, ω and their derivatives, and compute the partition function $Z(q,h) = \mathrm{Tr}(-1)^F q^{L_0} t^{J_0}$. (Similar counting of gauge invariant polynomials for the present and related models is given in [12].)

Weight 0. At the lowest level, the states are exhausted by

$$\lambda^{(\!(i_1} \cdots \lambda^{i_n)\!)} \,. \tag{3.7}$$

Here, the notation $(\!(i_1 \cdots i_n)\!)$ signifies the symmetric traceless tensor product. The states can be conveniently described using the Dynkin labels for $\mathrm{SO}(N) \times \mathrm{U}(1)_t$ as

$$(n00 \cdots 0)t^n \,. \tag{3.8}$$

Using the well-known dimension formulas for the symmetric tensors[5]

$$\dim(n00 \cdots 0) = \begin{cases} \prod_{i=2}^{k} \frac{(n+2k-i-1)(n+i-1)}{(2k-i-1)(i-1)} & \mathrm{SO}(2k)\,, \\ \prod_{i=2}^{k} \frac{(n-2k+1)(n+i-1)}{(2k-1)(i-1)} & \mathrm{SO}(2k+1)\,, \end{cases} \tag{3.9}$$

one gets

$$Z_0(t) = \sum_{n=0}^{\infty} \dim(n00 \cdots 0)t^n = \frac{1-t^2}{(1-t)^N} \,. \tag{3.10}$$

Note that the level 0 partition function is invariant under

$$\text{“field-antifield symmetry”}: \quad Z_0(t) = -(-t)^{2-N} Z_0(1/t) \,. \tag{3.11}$$

As explained in [10], the number $2-N$ on the exponent is the ghost number anomaly of the system. Since this symmetry plays an important role in our forthcoming discussions (as well as in the pure spinor superstring), let us explain the implication of its existence before going on to the weight 1 partition function.

Field-antifield symmetry. Suppose one couples the system to free fermionic bc systems $(p_i, \theta^i)_{i=1\sim N}$ of weight $(1,0)$, and extends the definition of the t-charge to the new sector as $t(p,\theta) = (-1,1)$. By an analogy with the pure spinor superstring, one also defines the “physical” BRST operator as

$$Q = \int \lambda^i p_i \,. \tag{3.12}$$

Then the symmetry $Z_0(t) = -(-t)^{2-N} Z_0(1/t)$ implies that all Q-cohomology elements appear in “spacetime” field-antifield pairs

$$V \ \text{ at } \ \pm t^n \quad \leftrightarrow \quad V_A \ \text{ at } \ \mp t^{2-n} \,. \tag{3.13}$$

Indeed, the total zero-mode partition function reads

$$\mathbf{Z}_0(t) = Z_{\lambda,0}(t) Z_{\theta,0}(t) = 1 - t^2 \,, \tag{3.14}$$

[5]Strictly speaking, those formulas are correct only for $k \geq 2$. Dimensions of symmetric traceless tensors for $N = 2,3$ are $\mathrm{SO}(2) = 2 - \delta_{n,0}$ and $\mathrm{SO}(3) = 2n+1$.

which is accounted for by a pair of "massless" cohomologies

$$1 \;\text{ at }\; t^0 \quad \leftrightarrow \quad (\lambda\theta) = \lambda^i \theta^i \;\text{ at }\; -t^2 \,. \tag{3.15}$$

The field-antifield symmetry implies the existence of a non-degenerate inner product that pairs every operator V to its antifield V_A

$$(V, V_A) = 1 \,. \tag{3.16}$$

For the case at hand, the inner product can be defined as the overlap

$$(V, W) = \lim_{z \to 0} \langle 0 | z^{2L_0} V(1/z) W(z) | 0 \rangle \,, \tag{3.17}$$

with the condition

$$\langle 0 | (\lambda\theta) | 0 \rangle = 1 \,. \tag{3.18}$$

It is easy to see that Q-exact states decouples from the inner product. Of course, this construction of the inner product is reminiscent of that of the pure spinor superstring [1] where one uses the rule

$$\langle 0 | (\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)(\lambda\gamma^\rho\theta)(\theta\gamma_{\mu\nu\rho}\theta) | 0 \rangle = 1 \,. \tag{3.19}$$

We will shortly observe that the space of gauge invariant polynomials at weight 1 and higher lacks the field-antifield symmetry. It might sound harmless but we stress the importance of having the field-antifield symmetry at all mass levels to define the "spacetime amplitude" appropriately. Otherwise, some "massive" vertex operators in the cohomology of $Q = \int \lambda^i p_i$ would unfavorably decouple from the amplitude. In fact, in the pure spinor formulation of superstring, demonstrating the existence of field-antifield symmetry for the full cohomology of $Q = \int \lambda^\alpha d_\alpha$ was an unresolved challenge. This and related issues will be reported in a separate communication [5].

Weight 1. Having explained the notion of field-antifield symmetry, let us go back to the construction of gauge invariant polynomials at weight 1. Here, one of $\partial\lambda$ or ω can be used to saturate the weight. $\partial\lambda$ must satisfy the constraint at level 1, $\partial(\lambda\lambda) = 2\lambda\partial\lambda = 0$, while the conjugate ω must appear in the combination which is invariant under the gauge transformation $\delta_\Lambda \omega = \Lambda\lambda$. At level 1, this condition implies that ω must appear in the form of the gauge invariant currents J and N_{ij}. Hence, the gauge invariant polynomials are $(n \geq 0)$

$$\begin{aligned}
\partial\lambda^{((j}\lambda^{i_1}\cdots\lambda^{i_n}))} &= (n+1, 00\cdots 0)t^{n+1} \,, \\
\partial\lambda^{[j}\lambda^{((k]}\lambda^{i_1}\cdots\lambda^{i_n}))} &= (n10\cdots 0)t^{n+2} \,, \\
\omega_j\lambda^{((j}\lambda^{i_1}\cdots\lambda^{i_n}))} &= (n00\cdots 0)t^{n} \,, \\
\omega^{[j}\lambda^{((k]}\lambda^{i_1}\cdots\lambda^{i_n}))} &= (n10\cdots 0)t^{n} \,.
\end{aligned} \tag{3.20}$$

Summing up the dimensions as before, one finds

$$Z_{1,\mathrm{poly}}(t) = \frac{Nt - t^2 - Nt^3 + t^4}{(1-t)^N} + \frac{\big(-1 + (1-t)^N\big)t^{-2} + Nt^{-1} + 1 - Nt}{(1-t)^N}\,. \tag{3.21}$$

The first term represents the contribution from $\partial\lambda$ and the second term represents that of ω.

Note that $Z_{1,\mathrm{poly}}(t)$ as defined in (3.21) does not posses the field-antifield symmetry. However, it is easy to see from the way we wrote it that

$$Z_1(t) = Z_{1,\mathrm{poly}}(t) - t^{-2} \tag{3.22}$$

satisfies the symmetry. This suggests that one needs an extra *fermionic* state with t-charge -2. In the BRST cohomology, this extra state corresponds to the ghost b. At first sight, there seems to be no room for fermionic states in the present setup, but in fact they can be employed as the elements of Čech-Dolbeault cohomologies at odd degrees.

Weight 2. Explicit constructions of the gauge invariant polynomials goes the same at the level 2.

First, there are polynomials with two ω's ($n \geq 0$):

$$\begin{aligned} N_{[\![i_1 i_2} N_{i_3 i_4]\!]}\lambda^{(n)} &= (\delta_{j_1[\![i_1}\omega_{i_2})(\omega_{i_3}\delta_{i_4]\!]j_2})\lambda^{(\!(j_1}\lambda^{j_2}\lambda^{k_1}\cdots\lambda^{k_n})\!)} = (n200\cdots 0)t^n\,,\\ N_{i_0 i_1}N^{i_0 i_2}\lambda^{(n)} &= (\delta_{i_0 j_1}\omega_{[\![i_1})(\delta^{i_0 j_2}\omega_{i_2]\!]})\lambda^{(\!(j_1}\lambda^{j_2}\lambda^{k_1}\cdots\lambda^{k_n})\!)} = (n+2,00\cdots 0)t^n\,,\\ N_{i_1 i_2}J\lambda^{(n)} &= (\delta_{j_1[i_1}\omega_{i2]})\omega_{j_2}\lambda^{(\!(j_1}\lambda^{j_2}\lambda^{k_1}\cdots\lambda^{k_n})\!)} = (n100\cdots 0)t^n\,,\\ JJ\lambda^{(n)} &= \omega_{j_1}\omega_{j_2}\lambda^{(\!(j_1}\lambda^{j_2}\lambda^{k_1}\cdots\lambda^{k_n})\!)} = (n00\cdots 0)t^n\,. \end{aligned} \tag{3.23}$$

Here, the symbol $[\![i_1 i_2\cdots i_n]\!]$ implies that the indices are traceless, block-symmetric, and antisymmetric within each blocks; in particular $[\![i_1, i_2]\!]$ simply denotes the traceless symmetric tensor.

Also, there is a gauge invariant function with negative t-charge:

$$\begin{aligned} f_i &= J\omega_i + N_{ij}\omega^j\\ &= -2(\lambda\omega)\omega_i + (\omega\omega)\lambda_i\,. \end{aligned} \tag{3.24}$$

In a local coordinate patch $U_1 = (g, u_a)$, components of f_i are given by $(v_a v_a)/g$ and its Lorentz transformations, both classically and quantum mechanically. Note, however, that polynomials of the form $f_i\,\lambda^{(n+1)}$ ($n \geq 0$) are not independent from the ones listed in (3.23).

As for the polynomials with a single derivative and a single ω, one finds the following independent states ($n \geq 0$):

$$\begin{aligned} N^{ij}\partial\lambda\lambda^{(n)} &= \big(\omega^{[i}\lambda^{(\!(j]}\partial\lambda^k\lambda^{k_1}\cdots\lambda^{k_n})\!)} + \omega^{[i}\partial\lambda^k\lambda^{(\!(j]}\lambda^{k_1}\cdots\lambda^{k_n})\!)}\\ &\qquad + \partial\lambda_i\omega^{[i}\lambda^{(\!(j]}\lambda^{k_1}\cdots\lambda^{k_n})\!)}\big) + \partial\lambda_{[k}\delta_{\ell]m}\omega^{[i}\lambda^{(\!(j]}\lambda^m\lambda^{k_1}\cdots\lambda^{k_n})\!)}\\ &= \big((n+1,10\cdots) + (n010\cdots) + (n+1,0\cdots)\big)t^{n+1} + (n20\cdots)t^{n+2}\,,\\ J\partial\lambda\lambda^{(n)} &= \omega_j\partial\lambda^{(\!(i}\lambda^j\lambda^{k_1}\cdots\lambda^{k_n})\!)} + \omega_k\partial\lambda^{[i}\lambda^{(\!(j]}\lambda^k\lambda^{k_1}\cdots\lambda^{k_n})\!)}\\ &= (n+1,0\cdots)t^{n+1} + (n10\cdots)t^{n+2}\,,\\ T &= \omega_i\partial\lambda^i = (00\cdots)t^0\,. \end{aligned} \tag{3.25}$$

Note that we could have included the energy momentum tensor T as the "$n = -1$ piece" of the $J\partial\lambda\lambda^{(n)}$ series; in other words, $T\lambda^{(n+1)}$ and $J\partial\lambda\lambda^{(n)}$ $(n \geq 0)$ are not independent.

Finally, there are two types of polynomials with two derivatives, $\partial^2\lambda\lambda^{(n)}$ and $(\partial\lambda)^2\lambda^{(n)}$, but some of them are related by the level 2 constraint

$$\lambda\partial^2\lambda + \partial\lambda\partial\lambda = 0\,. \tag{3.26}$$

A choice of independent polynomials are $(n \geq 0)$:

$$\begin{aligned}
\partial^2\lambda^i\lambda^{(\!(j_1}\cdots\lambda^{j_n)\!)} &= (10\cdots0)\otimes(n0\cdots0)t^{n+1}\,,\\
\partial\lambda^{(\!(i_1}\partial\lambda^{i_2}\lambda^{j_1}\cdots\lambda^{j_n)\!)} &= (n+2,0\cdots0)t^{n+2}\,,\\
\partial\lambda^{[i_1}\lambda^{(\!(j_1]}\partial\lambda^{j_2}\lambda^{j_2}\cdots\lambda^{j_n)\!)} &= (n+1,10\cdots0)t^{n+3}\,,\\
(\partial\lambda^{[\![i_1}\delta^{j_1}_{k_1})(\partial\lambda^{i_2}\delta^{j_2]\!]}_{k_2})\lambda^{(\!(k_1}\cdots\lambda^{k_n)\!)} &= (n20\cdots0)t^{n+4}\,.
\end{aligned} \tag{3.27}$$

Adding up all the contributions (3.23)$\sim$(3.25) and (3.27), one finds

$$\begin{aligned}
Z_{2,\text{poly}}(t) = {}& \frac{-N(t^{-3}-t^6)+\frac{(N+2)(N-1)}{2}(t^{-2}-t^5)+N(t^{-1}-t^3)+\frac{N^2-N+4}{2}(t^0-t^2)}{(1-t)^N}\\
& + Nt^{-3} + \frac{N^2-N+2}{2}t^{-2} + Nt^{-1}\,.
\end{aligned} \tag{3.28}$$

Again, $Z_{2,\text{poly}}(t)$ is non-invariant under the field-antifield symmetry, but the failure is modest:

$$\begin{aligned}
& Z_2(t) = Z_{2,\text{poly}}(t) - Nt^{-3} - \frac{N^2-N+2}{2}t^{-2} - Nt^{-1}\\
\rightarrow\quad & Z_2(t) = -(-t)^{2-N}Z_2(1/t)\,.
\end{aligned} \tag{3.29}$$

Classically, the elements of the BRST cohomology that correspond to the missing states are $b\omega_i$ at t^{-3}, bJ and bN_{ij} at t^{-2}, and $b\partial\lambda^i$ at t^{-1}, and one can construct the Čech cocycles corresponding to those states.

Quantum mechanically, there is a slight discrepancy in the interpretation of the symmetric partition function between the BRST and curved $\beta\gamma$ descriptions. That is, while both f_i and the Čech 1-cocycle corresponding to $b\partial\lambda^i$ are in the Hilbert space of the quantum curved $\beta\gamma$ description, both are not in the quantum BRST cohomology, as they form a BRST doublet (with an exception of the $N = 6$ model). Note, however, that both descriptions still lead to the same symmetric partition function: f_i and $b\partial\lambda^i$ have same charges except for the statistics so even classically they do not give a net contribution to the partition function $\mathrm{Tr}[(-1)^F\cdots]$.

3.3 BRST cohomology and symmetries of partition function

Since the BRST operator D carries t-charge 0, the partition function of D-cohomology coincides with that of the unconstrained space of (ω, λ, b, c) in which the cohomology is computed. This is because the elements not in the cohomology form BRST doublets and cancel out due to $(-1)^F$. Therefore, the partition function is simply given by [13]

$$Z(q,t) = \frac{1-t^2}{(1-t)^N} \prod_{h=1}^{\infty} \frac{(1-t^2q^h)(1-t^{-2}q^h)}{(1-tq^h)^N(1-t^{-1}q^h)^N} \,. \tag{3.30}$$

By expanding in q, partition functions at fixed Virasoro levels can be readily obtained.

The full partition function enjoys the following two symmetries, which turn out to be of fundamental importance. First is the "field-antifield symmetry" we already encountered:

$$Z(q,t) = -(-t)^{2-N} Z(q,1/t) \,. \tag{3.31}$$

As explained above, this symmetry is important to have a nice inner product after coupling to the fermionic partners (p_i, θ^i). The other is what we shall call "$*$-conjugation symmetry"

$$Z(q,q/t) = -q^1 t^{-2} Z(q,q/t) \,. \tag{3.32}$$

A little computation shows that this symmetry relates the states at $q^m t^n$ and those at $q^{1+m+n}t^{-2-n}$, which suggests the existence of an inner product pairing those. The inner product is constructed in section 4.1 using a conjugation operation $*$, which is a generalization of the standard BPZ conjugation [16].

Although not apparent at this stage, the inner product responsible for the $*$-conjugation symmetry turns out to be useful for probing the structure of the BRST cohomology $H^*(D)$, because it pairs the states with charges

$$q^m t^n g^k \quad \leftrightarrow \quad q^{1+m+n} t^{-2-n} g^{1-k} \,. \tag{3.33}$$

(The exponent of g indicates the ghost number.) This implies that the elements of $H^k(D)$ and $H^{1-k}(D)$ appear in pairs, and we utilize this information to show that the cohomology is non-vanishing only at ghost numbers 0 and 1 (see section 4).

Since $H^0(D)$ is equivalent to the space of gauge invariant polynomials, the missing states we found above should be contained in $H^1(D)$. We now explicitly confirm this statement at several lower mass levels.

3.4 Extra states in BRST cohomology

In the previous two subsections, we found that the partition function of the BRST cohomology possesses the field-antifield symmetry while that of the gauge invariant polynomials does not. We here explicitly construct the elements of the BRST cohomology and identify the extra states that are responsible for the discrepancy.

Weight 0: the zero mode contributions to the full partition function (3.30) is simply

$$Z_0(t) = \frac{1-t^2}{(1-t)^N} \,, \tag{3.34}$$

and it coincides with the result obtained from counting the number of gauge invariant polynomials (3.10). Indeed, since functions of the form $cf(\lambda)$ are never D-closed, and since the functions of the form $(\lambda\lambda)f(\lambda)$ are D-exact, cohomology representatives can be taken as

$$\lambda^{(\!(i_1} \cdots \lambda^{i_n)\!)}\,, \tag{3.35}$$

but now with λ's unconstrained. Of course, this is expected from the outset as the BRST construction is designed to realize what we have just described.

Weight 1: from (3.30) one immediately finds

$$Z_1(t) = \frac{-t^{-2} + Nt^{-1} + 1 - t^2 - Nt^3 + t^4}{(1-t)^N}\,, \tag{3.36}$$

and it possesses the field-antifield symmetry unlike the level 1 partition function $Z_{1,\mathrm{poly}}(t)$ of the gauge invariant polynomials. As expected, $Z_1(t)$ contains an extra fermionic state with respect to $Z_{1,\mathrm{poly}}(t)$:

$$Z_1(t) - Z_{1,\mathrm{poly}}(t) = -t^{-2}\,. \tag{3.37}$$

Clearly, the cohomology element responsible for $-t^{-2}$ is the BRST ghost

$$b\,, \quad \text{carrying charges } -q^1 t^{-2} g^1\,. \tag{3.38}$$

This state is paired with $\mathbf{1}$ at $q^0t^0g^0$ under the $*$-conjugation symmetry. The remaining states correspond to the gauge invariant polynomials (3.20). Cohomology representatives basically take the same form, but for $\omega_{i_1}\lambda^{(\!(i_1} \cdots \lambda^{i_n)\!)}$ it is given by replacing

$$-\omega\lambda \quad \to \quad J_t = -\omega\lambda - 2bc\,. \tag{3.39}$$

To summarize, weight 1 cohomology consists of $H^0(D)|_{h=1}$ (gauge invariant polynomials) and a single state b from $H^1(D)|_{h=1}$. Note that this is completely consistent with the structure expected from the $*$-conjugation symmetry. (Gauge invariant states with higher t-charges are paired with states with higher weights and $\mathbf{1}$ is the only operator which has the partner in the weight 1 sector.)

Weight 2: the analysis at weight 2 is similar. The partition function respects the field-antifield symmetry and reads

$$Z_2(t) = \frac{-N(t^{-3}-t^6) + \frac{(N+2)(N-1)}{2}(t^{-2}-t^5) + N(t^{-1}-t^3) + \frac{N^2-N+4}{2}(t^0-t^2)}{(1-t)^N}\,. \tag{3.40}$$

The extra states contained are[6]

$$Z_2(t) - Z_{2,\mathrm{poly}}(t) = -Nt^{-3} - \frac{N^2-N+2}{2}t^{-2}\,, \tag{3.41}$$

[6]Here, we removed from $Z_{2,\mathrm{poly}}$ the polynomial f_i at t^{-1} since it is not in the quantum BRST cohomology, as explained above. Classically, one would add $0 = (N-N)t^{-1}$ (f_i and $b\partial\lambda^i$) on the right hand side.

and one can check that those corresponds to

$$(b\omega_i,\ bJ,\ bN_{ij}) \quad \big(\overset{*}{\longleftrightarrow} \quad (\lambda^i, J, N_{ij})\big)\,. \tag{3.42}$$

Again, those states all carry ghost number 1.

At this point, the pattern of the pairing between $H^0(D)$ and $H^1(D)$ should have become clear. That is, whenever one has a ghost number 0 cohomology $F(\omega,\lambda,J,N;\partial)$ (gauge invariant polynomial), the corresponding ghost number 1 cohomology is obtained basically by swapping ω and λ, and multiplying b:

$$bF(\lambda,\omega,J,N;\partial) \quad \overset{*}{\longleftrightarrow} \quad F(\omega,\lambda,J,N;\partial)\,. \tag{3.43}$$

Although the precise representatives for $H^1(D)$ in general contain terms other than bF, one can check that the mapping (3.43) is consistent with the $*$-conjugation symmetry.

3.5 Remark on $\lambda^2 = 0$ model ($N = 1$)

Let us make a digression and make a comment on the $N = 1$ model. As mentioned earlier, the constraint for the seemingly simple model $\lambda^2 = 0$ is in fact reducible and the use of the naive BRST operator $D = \int b\lambda^2$ cannot be justified. Although D is nilpotent and it makes sense to consider its cohomology, the cohomology contains unwanted states outside ghost numbers 0 and 1. Let us explicitly identify some unwanted states which are the artifact of the improper application of the BRST method.

The full partition function of the D-cohomology is given by

$$Z(q,t) = \frac{1-t^2}{(1-t)}\prod_{h=1}^{\infty}\frac{(1-t^2q^h)(1-t^{-2}q^h)}{(1-tq^h)(1-t^{-1}q^h)}\,, \tag{3.44}$$

and it possess the two symmetries

$$Z(q,t) = t^1 Z(q,1/t)\,, \quad Z(q,t) = -q^1t^{-2}Z(q,q/t)\,. \tag{3.45}$$

At levels 0 and 1, the partition functions read

$$\begin{aligned} Z_0(t) &= 1+t\,,\\ Z_1(t) &= -t^{-2}+1+t-t^3\,. \end{aligned} \tag{3.46}$$

It is easy to obtain the cohomology representatives responsible for the partition functions. As usual, $-q^1t^{-2}$ corresponds to b and all others but the state at $-q^1t^3$ correspond to some gauge invariant polynomials.

However, the fermionic state at $-q^1t^3$ is found to be an unwanted state

$$(-2c\partial\lambda + \partial c\lambda)\,, \tag{3.47}$$

carrying ghost number -1. As can be seen from the naive relation $c \sim \lambda^2$, the occurrence of this state is related to the fact that the constraint $G \equiv \lambda^2 = 0$ and its derivative are not independent:

$$2G\partial\lambda = \partial G\lambda\,. \tag{3.48}$$

(In the standard BRST procedure, one would introduce a pair of bosonic ghost-for-ghost and extend the BRST operator D to kill this state.)

Finally, let us identify the state paired with $(-2c\partial\lambda + \partial c\lambda)$ under the $*$-conjugation symmetry $q^m t^n \leftrightarrow q^{m+n+1} t^{-2-n}$. The conjugate is at $q^5 t^{-5}$ which is the first term of the level 5 partition function

$$Z_5(t) = t^{-5} - 3t^{-3} - 5t^{-2} + 7 + 7t - 5t^3 - 3t^4 + t^6 \,. \tag{3.49}$$

The fact that the state at $q^5 t^{-5}$ is *bosonic* already implies that it is an unwanted state, since it necessarily carries even ghost number (which can easily be shown to be non-zero). The state is

$$b\partial b\partial\omega \simeq b\partial^2 b\omega \quad (\text{at } q^5 t^{-5} g^2) \tag{3.50}$$

carrying ghost number 2. For $N \geq 2$ models, one can show that both $b\partial b\partial\omega_i$ and $b\partial^2 b\omega_i$ are trivial, but for $N = 1$ (with the "wrong" BRST operator) only a linear combination of them is trivial.

4. Structure of quantum BRST cohomology

In the previous section, we compared the partition function of gauge invariant polynomials and that of the BRST cohomology, and found some extra states in the latter. This is not strange. The BRST construction relates the ghost number 0 cohomology to the space of gauge invariant polynomials, but there in general can be cohomologies at non-zero ghost numbers. In this and the next sections, we study those extra states in more detail. First, in this section, we show (for models with $N \geq 2$) that the quantum BRST cohomology is non-vanishing only at ghost numbers 0 and 1, and that the states in the two sectors come in pairs. Then in the next section, we explain how the ghost number 1 states can be described in the Čech or Dolbeault formalisms.

4.1 Inner product

In order to show that the cohomology elements come in pairs, we first define an inner product in the space $\mathcal{F}$ of all operators (not necessarily in the cohomology). Our inner product is a generalization of the standard BPZ inner product [16], and it is non-degenerate in the sense

$$\forall_{W\in\mathcal{F}} \ \langle V, W\rangle = 0 \quad \to \quad V = 0\,. \tag{4.1}$$

In other words, every non-zero operator V (not necessarily in the cohomology) should have at least one operator W satisfying $\langle V, W\rangle \neq 0$.

Let us denote the SL_2 invariant vacuum as

$$\mathbf{1} \ \sim \ |\mathbf{1}\rangle = |0\rangle\,. \tag{4.2}$$

In the present case, the vacuum satisfies

$$b_n|0\rangle = \omega_{i,n}|0\rangle = 0\,,\ (n \geq 0)\,, \quad c_n|0\rangle = \lambda^i_n|0\rangle = 0\,, \quad (n \geq 1)\,, \tag{4.3}$$

where as usual the mode expansion of a weight h primary field is

$$\phi(z) = \sum_n \phi_n z^{-n-h}\,. \tag{4.4}$$

The "in states" are constructed by acting the creation operators $(b_{-n-1}, c_{-n}, \omega_{-n-1}, \lambda_{-n})_{n\geq 0}$ on the vacuum $|0\rangle$. Using the state-operator mapping, in states can also be described as

$$|V\rangle = \lim_{z\to 0} V(z)|0\rangle\,, \tag{4.5}$$

for some operator V which is a polynomial of b, c, ω, λ and their derivatives.

Bosonizing the bosonic $\beta\gamma$ fields as $(\beta_i, \gamma_i) = (\partial\xi_i \mathrm{e}^{-\phi_i}, \mathrm{e}^{\phi_i}\eta_i)$ [17] and setting $\phi = \sum_i \phi_i$, the "out states" are constructed using the conjugate operation $*$ defined by

$$\langle V| = |V\rangle^* \quad \begin{cases} |0\rangle^* = \langle\Omega| = \langle 0|\mathrm{e}^{-\phi} c_0 c_1\,, \\ b_n^* = b_{-n-2}\,, \quad c_n^* = c_{-n+2}\,, \quad \omega_{i,n}^* = \omega_{i,-n-1}\,, \quad {\lambda_n^i}^* = \lambda_{-n+1}^i\,. \end{cases} \tag{4.6}$$

In terms of conformal fields, those can be described as a modified BPZ conjugate state with $\mathrm{e}^{-\phi}c\partial c$ inserted at infinity:

$$\langle V| = \lim_{z\to\infty} \langle \mathrm{e}^{-\phi} c\partial c| z^{2L_0+J_0} V(z)\,. \tag{4.7}$$

Now, we define the inner product by the overlap of Fock states

$$\langle V, W\rangle = \langle V|W\rangle \tag{4.8}$$

with the rule (recall $\langle\Omega| = |0\rangle^*$)

$$\langle\Omega|b_{-1}|0\rangle = 1\,. \tag{4.9}$$

Equivalently, using the notation of conformal field theory, it can be defined as

$$\langle V, W\rangle = \lim_{z\to\infty, w\to 0} z^{2L_0+J_0} \langle\!\langle V(z)W(w)\rangle\!\rangle\,, \tag{4.10}$$
$$\text{where} \quad \langle\!\langle V(z)W(w)\rangle\!\rangle = \langle \mathrm{e}^{-\phi}c\partial c|V(z)W(w)|0\rangle\,.$$

Since we inserted $\mathrm{e}^{-\phi}c\partial c$ at the infinity, the rule is consistent with the standard rule expected from anomalies, i.e. $\langle 0|\mathrm{e}^{-\phi}c_0|0\rangle = 1$.

4.2 Pairing of cohomology

Up to this point, our argument was general and had nothing to do with the BRST structure of the system. We now turn to discuss the implication of the inner product on the BRST cohomology. First, since $D(\mathrm{e}^{-\phi}c\partial c) = 0$, the BRST trivial operators decouple from the inner product (4.10). Therefore,

$$\langle\!\langle D(VW)\rangle\!\rangle = 0 \quad \leftrightarrow \quad \langle DV, W\rangle + \langle V, DW\rangle = 0\,. \tag{4.11}$$

Using this property, it is easy to show that the cohomology elements come in pairs.

Let us split the space of operators $\mathcal{F}$ as follows:

$$\mathcal{F} = \mathcal{A} + \mathcal{B} + \mathcal{H} = \begin{cases} \mathcal{A}: & D\text{-non-closed}\,, \\ \mathcal{B}: & D\text{-exact}\,, \\ \mathcal{H}: & D\text{-cohomology}\,. \end{cases} \tag{4.12}$$

Although there is no canonical way to achieve the splitting between $\mathcal{B}$ and $\mathcal{H}$, one can argue that the inner product (4.10) induces a non-degenerate inner product on the cohomology $\mathcal{H}$. This follows from the following two properties:

1. $V \in \mathcal{B}$ and $\langle V, W\rangle \neq 0 \rightarrow W \in \mathcal{A}$ $(DW \neq 0)$
2. $V \in \mathcal{A} \rightarrow \exists W \in \mathcal{B}$ s.t. $\langle V, W\rangle \neq 0$

Proof of 1. Let V_c denote a conjugate of $V \in \mathcal{F}$, i.e. $\langle V, V_c\rangle \neq 0$. (It is not unique but we do not rely on the uniqueness of V_c in the following arguments.) Since V is D-exact, it can be written as $V = DU$ for some U. For all V_c, one has

$$0 = \langle\!\langle D(UV_c)\rangle\!\rangle = \langle\!\langle (DU)V_c\rangle + \langle\!\langle U(DV_c)\rangle\!\rangle\,, \tag{4.13}$$

but since $\langle\!\langle (DU)V_c\rangle = \langle\!\langle VV_c\rangle \neq 0$, it follows that $\langle\!\langle U(DV_c)\rangle\!\rangle \neq 0$ which in turn implies $DV_c \neq 0$ (and $U_c = DV_c$).

Proof of 2. Denote $U \equiv DV \neq 0$ and let U_c be one of its conjugate. Then,

$$0 = \langle\!\langle D(U_cV)\rangle\!\rangle = \langle\!\langle (DU_c)V\rangle + \langle\!\langle U_c(DV)\rangle\!\rangle\,, \tag{4.14}$$

and since $\langle\!\langle U_c(DV)\rangle\!\rangle = \langle\!\langle U_cU\rangle \neq 0$, one finds $V_c = DU_c$.

Now, the property 1 implies $\langle \mathcal{B}, \mathcal{B}\rangle = \langle \mathcal{B}, \mathcal{H}\rangle = 0$, while the property 2 implies that the matrix $\langle \mathcal{A}, \mathcal{B}\rangle$ has the maximal rank. Thus, schematically, the inner product for the full space looks like the first matrix in the diagram below. (The star $\star$ signifies the maximal rank and the question mark ? designates blocks whose properties are unknown.) This then implies that one can choose appropriate representatives for the cohomology $\mathcal{H}$ so that $\langle \mathcal{A}, \mathcal{H}\rangle = 0$ (the second matrix). Finally, the non-degeneracy of the submatrix $\langle \mathcal{H}, \mathcal{H}\rangle$ follows from that of the full matrix.

$$\begin{array}{cc} & \begin{array}{ccc} \mathcal{A} & \mathcal{B} & \mathcal{H} \end{array} \\ \begin{array}{c} \mathcal{A} \\ \mathcal{B} \\ \mathcal{H} \end{array} & \begin{pmatrix} ? & \star & ? \\ \star & 0 & 0 \\ ? & 0 & ? \end{pmatrix} \end{array} \rightarrow \begin{array}{cc} & \begin{array}{ccc} \mathcal{A} & \mathcal{B} & \mathcal{H} \end{array} \\ \begin{array}{c} \mathcal{A} \\ \mathcal{B} \\ \mathcal{H} \end{array} & \begin{pmatrix} ? & \star & 0 \\ \star & 0 & 0 \\ 0 & 0 & ? \end{pmatrix} \end{array} \rightarrow \begin{array}{cc} & \begin{array}{ccc} \mathcal{A} & \mathcal{B} & \mathcal{H} \end{array} \\ \begin{array}{c} \mathcal{A} \\ \mathcal{B} \\ \mathcal{H} \end{array} & \begin{pmatrix} ? & \star & 0 \\ \star & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \quad (\because \det \neq 0)\,.$$

4.3 Vanishing theorem for $H^k(D)$ with $k \neq 0, 1$

Using the pairing of cohomologies just described, one can show that the BRST cohomology is non-vanishing only at ghost numbers 0 and 1. To see this, recall that the quantum charges of a state and its $*$-conjugate are related as

$$q^m t^n g^k \quad \leftrightarrow \quad q^{m+n+1} t^{-2-n} g^{1-k}\,, \tag{4.15}$$

where m is the weight, n is the t-charge, and k is the ghost number. Our claim is then equivalent to the assertion $H^k(D) = 0$ $(k < 0)$. That is, there are no cohomology elements with negative ghost numbers (which means the number of c ghosts is strictly greater than that of b ghosts). $H^k(D) = 0$ $(k < 0)$ is true more or less by construction, but let us briefly sketch why it is the case.

In the BRST formalism, c-type ghosts represent the constraint ($c \stackrel{D}{\to} \lambda\lambda$) and the formalism is designed so that the c-type ghosts do not contribute to the cohomology in any important way. By construction, there are no negative ghost number cohomologies without b; whenever there is a D-closed operator of the form

$$f_k(\omega, \lambda, c; \partial) = \sum_{\{N\}} \partial^{N_1} c \cdots \partial^{N_k} c \, f_{N_1 \cdots N_k}(\omega, \lambda; \partial)\,, \tag{4.16}$$

one can show that it is D-exact. (If this is not the case, additional c-type ghosts must be introduced and the BRST charge must be extended to make it D-exact, $c' \stackrel{D}{\to} f_k$. This will be the case when the constraints are reducible.) In fact, it can be shown that the same is true for the negative ghost number operators with both b and c [18],

$$f_k(\omega, \lambda, b, c; \partial) = \sum_{i \geq 0} \sum_{\{M,N\}} \partial^{M_1} b \cdots \partial^{M_i} b \, \partial^{N_1} c \cdots \partial^{N_{k+i}} c \, f_{M_1 \cdots M_i N_1 \cdots N_{k+i}}(\omega, \lambda; \partial)\,. \tag{4.17}$$

If f_k is D-closed, the terms without b $(i = 0)$ can be written in a D-exact form, modulo terms with at least one b $(i \geq 1)$. After subtracting the D-exact piece just mentioned, the equation $Df_k = 0$ implies that the coefficients of $\partial^{M_1} b$, i.e. $\partial^{N_1} c \cdots \partial^{N_{k+1}} c \; f_{M_1 N_1 \cdots N_{k+1}}$, are D-closed (and hence D-exact) modulo terms with at least two b's. Therefore, f_k is D-exact modulo terms with at least two b's $(i \geq 2)$. Proceeding inductively in number of b's, one can show that f_k is D-exact.

Therefore, we conclude that $H^k(D) = 0$ $(k < 0)$, and hence $H^k(D) = 0$ $(k > 1)$ via the $*$-conjugation symmetry.

5. Relating BRST, Čech and Dolbeault cohomologies

In the previous section, we found that the BRST cohomology includes extra states at ghost number 1 that do not correspond to gauge invariant polynomials. Those states were important for having the field-antifield symmetry. We here sketch the equivalence between the BRST and Čech/Dolbeault descriptions, by giving a mapping that relates the classical pieces of the cohomology element. In particular we shall explain how the ghost number 1 extra states are described in the intrinsic Čech/Dolbeault framework.

Since the BRST and the intrinsic curved $\beta\gamma$ frameworks use different normal ordering prescriptions, the quantum BRST and Čech-Dolbeault cohomologies differ in general. This indeed happens for our models. However, as we have mentioned several times, our partition function $\mathrm{Tr}[(-1)^F \cdots]$ is insensitive to such discrepancies.

5.1 BRST, Čech and Dolbeault cohomologies

It is convenient to introduce the following four cohomologies, which classically give different representation of a same space:

1. Minimal BRST: Cohomology of D
2. Non-minimal BRST: Cohomology of $D + \overline{\partial}_X$
3. Dolbeault cohomology $\overline{\partial}_X$ (of gauge invariant operators)
4. Čech cohomology (of gauge invariant operators)

As explained in 2.2, the notion of "gauge invariance" in curved $\beta\gamma$ frameworks (for $N \geq 4$ models) is a simple way to refer to the operator intrinsic to the target space X but by using the extrinsic coordinate (ω, λ) of the space where X is embedded. We find it especially useful when comparing to the BRST framework.

Although we already described most of them, let us recapture the definitions of each.

Minimal BRST cohomology. This is simply the standard BRST cohomology of $D = \int b(\lambda\lambda)$, computed in the space of polynomials of unconstrained (ω, λ), BRST ghosts and their derivatives,

$$f(\omega, \lambda, b, c; \partial)\,. \tag{5.1}$$

By construction, the ghost number 0 cohomology $H^0(D)$ is isomorphic to the space of gauge invariant polynomials of the constrained system. On the other hand, as we observed above, there are also the operators with non-zero ghost numbers, but the higher cohomology is non-empty only at ghost number 1 (where b carries ghost number $+1$). Obtaining the expressions for those extra states in the curved $\beta\gamma$ framework, i.e. in the Čech/Dolbeault cohomologies, is the goal of the present section.

Non-minimal BRST cohomology. Closely related to the minimal BRST cohomology is what we call non-minimal BRST cohomology. This is defined by introducing the *unconstrained* non-minimal variables $(\overline{\omega}^i, \overline{\lambda}_i; s^i, r_i)$ and extending the BRST operator as

$$\overline{D} = D + \overline{\partial}_X\,, \quad \overline{\partial}_X = -r_i\overline{\omega}^i \sim \mathrm{d}\overline{\lambda}_i \frac{\partial}{\partial\overline{\lambda}_i}\,. \tag{5.2}$$

The cohomology of $\overline{D}$ is computed in the space of functions of the form

$$f(\omega, \lambda, \overline{\omega}, \overline{\lambda}, r, s, b, c; \partial)\,, \tag{5.3}$$

where now f can diverge as fast as $(\lambda\overline{\lambda})^{-n}$ for $n < N$.

The restriction on the order of poles is important. If one allows the functions that diverge as fast as $(\lambda\overline{\lambda})^{-N}$, there will be extra cohomology elements due to the operator

$$\frac{\overline{\lambda}_{[i_1} r_{i_2} \cdots r_{i_N]}}{(\lambda\overline{\lambda})^N}\,, \tag{5.4}$$

which do not have counterparts in minimal BRST cohomology.

We introduced the non-minimal variables as unconstrained variables, however, it should be noted that they do not affect the cohomology even if they are considered to be constrained, as long as the minimal variables are *unconstrained*. Whether constrained or not, the non-minimal variables can appear only in the combinations $\overline{\lambda}_i(\lambda\overline{\lambda})^{-1}$ and $r_i(\lambda\overline{\lambda})^{-1}$ (other combinations of non-minimal variables are irrelevant due to the usual quartet mechanism), and one can switch between the two viewpoints by simply forgetting/imposing the non-minimal constraint.

Non-minimal BRST description is a hybrid between minimal BRST and Dolbeault languages, and provides the key to relate the minimal BRST and Dolbeault descriptions. The space on which $\overline{D}$ acts (5.3) is doubly graded by the BRST ghost number and the Dolbeault form degree.

Dolbeault cohomology. We now turn to the description of cohomologies in the curved $\beta\gamma$ schemes. The cohomology of the differential operator $\overline{\partial}_X = -r_i\overline{\omega}^i$ is computed in the space of functions of the form

$$f(\omega, \lambda, \overline{\omega}, \overline{\lambda}, r, s; \partial)\,. \tag{5.5}$$

Again, f is allowed to diverge as $(\lambda\overline{\lambda})^{-n}$ $(n < N)$, but additionally it must be gauge invariant (if one is to write f using the extrinsic coordinates (ω, λ)).

The cohomology splits naturally into two families. One family is the globally defined gauge invariant polynomials without poles in $(\lambda\overline{\lambda})$. The other corresponds to the operators in the higher BRST cohomology. The BRST ghost number corresponds to the form degree of the Dolbeault cohomology, i.e. the number of r_i's (that can only appear in the combination $(\lambda\overline{\lambda})^{-1}r_i$). Since operators diverging too fast as $(\lambda\overline{\lambda}) \to 0$ are troublesome for the computation of amplitudes [3], we do not want to have cohomologies at too high degrees.

Čech cohomology. Finally, the Čech-type description of the cohomology is obtained from the Dolbeault description using the usual Čech-Dolbeault correspondence. Elements of the cohomology will be the Čech n-cocycles of the form

$$f = (f^{A_0\cdots A_n}) = f^{A_0\cdots A_n}(\omega, \lambda; \partial)\,, \quad (n \geq 0)\,, \tag{5.6}$$

where $f^{A_0\cdots A_n}$ denotes a collection of gauge invariant functions defined on overlaps $U_{A_0\cdots A_n} = U_{A_0} \cap \cdots \cap U_{A_n}$. On $U_{A_0\cdots A_n}$, f is allowed to have poles in λ^{A_i} $(i = 0 \sim n)$. The degrees of cochains are related to the form degree in Dolbeault description, and hence to the BRST ghost numbers. The gauge invariant polynomials are represented as 0-cocycles, and the extra states at ghost number n are represented as n-cocycles that are defined modulo n-coboundaries.

5.2 Classical equivalence of various cohomologies

Operators in the four cohomologies in the previous subsection can be related as indicated in the following figure.

$$\begin{tikzcd}
\text{minimal BRST} \arrow[d, leftrightarrow, "(a)"] & & \text{Čech} \arrow[d, leftrightarrow, "(d)"] \\
\text{non-minimal BRST} \arrow[loop below, "(b)"] \arrow[rr, shift right, "(c')"'] & & \text{Dolbeault} \arrow[ll, shift right, "(c)"']
\end{tikzcd}$$

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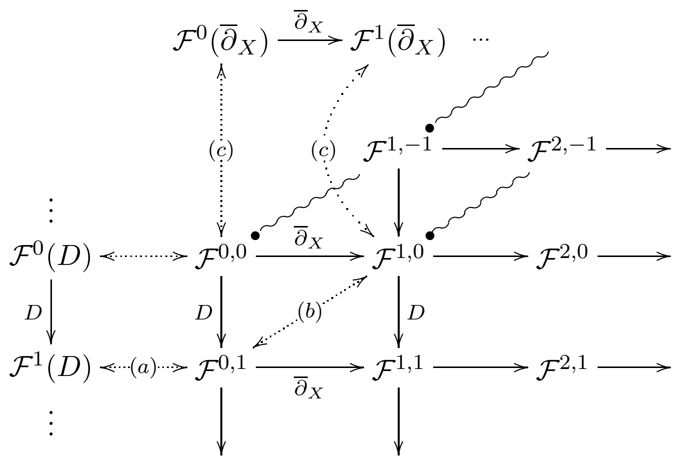

Figure 1: Embedding to the non-minimal BRST cohomology.

(a) Adding/removing non-minimal quartet under $\overline{\partial}_X = -r\overline{\varpi}$

(b) Different choice of cohomology representatives

(c) Embedding to “extrinsic” space of free fields

(c') Restriction to “intrinsic” (or gauge invariant) operators on X

(d) Standard Čech-Dolbeault mapping (partition of unity)

The idea here is to use the non-minimal BRST cohomology $H^*(D+\overline{\partial}_X)$ to bridge between the BRST and curved $\beta\gamma$ schemes, as figure 1 indicates.

In the figure, we put the minimal D-cohomology on the left-most column and the $\overline{\partial}_X$-cohomology on the top row. The non-minimal BRST cohomology of $(D+\overline{\partial}_X)$ is graded by the sum of BRST ghost number and the Dolbeault form degree (number of r's), which runs diagonally from north-west to south-east.

Both D and $\overline{\partial}_X$ cohomologies can be embedded in the $(D+\overline{\partial}_X)$-cohomology as indicated by the arrows (a) and (c). A ghost number k element of the D-cohomology can be regarded as a $(D+\overline{\partial}_X)$-cohomology element with degree $(0,k)$. A degree n element of the $\overline{\partial}_X$-cohomology can also be regarded as a $(D+\overline{\partial}_X)$-cohomology element, but this time the corresponding element in general has multiple (bi)degrees $\sum_{k\geq 0}\mathcal{F}^{n+k,-k}$.

Once the embedding into the non-minimal $(D+\overline{\partial}_X)$-cohomology is achieved, the cohomologies of D and $\overline{\partial}_X$ simply correspond to different choices of cohomology representatives, where the non-minimal variables are absent (minimal BRST), and the (b-type) BRST ghosts are absent (Dolbeault), as indicated by the arrow (b).

5.2.1 Embedding to non-minimal BRST cohomology

Embedding (a). First, let us describe the embedding of the minimal BRST cohomology to the non-minimal BRST cohomology. Since D and $\overline{\partial}_X$ anticommute, cohomology of $\overline{D}$ is the cohomology of D computed in the cohomology of $\overline{\partial}_X$. Note that the $\overline{\partial}_X$-cohomology

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here is computed in the space where the constraint for the minimal variable λ is absent. Hence, provided one restricts the order of poles in $(\lambda\overline{\lambda})$, the cohomology of $\overline{\partial}_X$ is simply the space without non-minimal variables. That is, all elements of the $\overline{\partial}_X$-cohomology have representatives of the form

$$f(\omega, \lambda, b, c; \partial) \quad \text{(no poles in } \lambda) \,, \tag{5.7}$$

which is nothing but the space where the minimal BRST cohomology is computed.

Embedding (*c*). For the models at hand, a Dolbeault cohomology element with form degree n can be represented by a gauge invariant function f^n. Classically, from f^n, one gets an operator $f^{n,0}$ living in the space $\mathcal{F}^{n,0}$, by simply forgetting the constraint $(\lambda\lambda) = 0$. In contrast to the elements of the minimal BRST cohomologies above, however, $f^{n,0}$ is not necessarily $(D + \overline{\partial}_X)$-closed. Nevertheless, following the standard argument in the BRST formalism, $f^{n,0}$ can be extended to the form $\hat{f}^n = \sum_{k\geq 0} \hat{f}^{n+k,-k}$ so that

$$(D + \overline{\partial}_X)\hat{f} = 0 \quad \Leftrightarrow \quad \left\{ \begin{aligned} D\hat{f}^{n,0} &= 0\,, \\ D\hat{f}^{n+1,-1} + \overline{\partial}_X \hat{f}^{n,0} &= 0\,, \\ &\;\vdots \\ D\hat{f}^{n+p-1,-p+1} + \overline{\partial}_X \hat{f}^{n+p-2,-p+2} &= 0\,, \\ \overline{\partial}_X \hat{f}^{n+p,-p} &= 0\,, \end{aligned} \right. \tag{5.8}$$

for some p, or, more pictorially,

$$\begin{tikzcd}
& \hat{f}^{n,0} \arrow[dl, "D"'] \arrow[dr, "\overline{\partial}_X"] & + & \hat{f}^{n+1,-1} \arrow[dl, "D"'] \arrow[dr, "\overline{\partial}_X"] & + \;\cdots & + & \hat{f}^{n+p,-p} \arrow[dl, "D"'] \arrow[dr, "\overline{\partial}_X"] & \\
0 & & 0 & & 0 & 0 & & 0
\end{tikzcd}$$

That is, a Dolbeault cohomology element with degree n corresponds to a sequence of non-minimal operators with its "head" in $\mathcal{F}^{n,0}$ (see figure 1).

For completeness, let us briefly sketch the procedure to obtain the sequence $\hat{f}^n = \sum_{k\geq 0} \hat{f}^{n+k,-k}$, starting from a constrained operator f^n. Firstly, the unconstrained operator $f^{n,0}$ naively obtained from f^n is not necessarily D-closed, but it satisfies

$$\begin{aligned} Df^{n,0} &= g^{n,1} \approx 0 \quad \text{(gauge invariance of } f^n)\,, \\ \overline{\partial}_X f^{n,0} &= g^{n+1,0} \approx 0 \quad (\overline{\partial}_X\text{-closed condition of } f^n)\,, \end{aligned} \tag{5.9}$$

for some $g^{n,1} \in \mathcal{F}^{n,1}$ and $g^{n+1,0} \in \mathcal{F}^{n+1,0}$. As indicated in the first formula, gauge invariance of the original f^n implies that $g^{n,1}$ vanishes on $(\lambda\lambda) = 0$, and of course $g^{n,1}$ contains one b. Hence, $g^{n,1}$ can be written as $D\bar{f}^{n,0}$ where $\bar{f}^{n,0}$ is different from $f^{n,0}$. For example, for $f^{n,0} = \lambda\omega$, one has $\overline{\partial}_X f^{0,0} = 0$ and

$$Df^{0,0} = g^{0,1} = 2b(\lambda\lambda) = D\bar{f}^{0,0} \quad \text{where} \quad \bar{f}^{0,0} = \frac{(\overline{\lambda}\omega)(\lambda\lambda)}{\lambda\overline{\lambda}}\,. \tag{5.10}$$

By setting $\hat{f}^{n,0} = f^{n,0} - \bar{f}^{n,0}$, one obtains the "head" of the chain $\hat{f}^n$ in (5.8).

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On the other hand, using $\{\overline{\partial}_X\,, D\} = D^2 = 0$ and the second equation in (5.9), one finds after a little computation that

$$\overline{\partial}_X \hat{f}^{n,0} = \hat{g}^{n+1}\,(\equiv g^{n+1,0} - \bar{g}^{n+1,0})\,, \tag{5.11}$$

where $g^{n+1,0} = \overline{\partial}_X f^{n,0}$ and $\bar{g}^{n+1,0} = \overline{\partial}_X \bar{f}^{n,0}$ are separately D-closed. In fact both are weakly zero and hence are D-exact. For example, $\bar{f}^{0,0}$ in (5.10) satisfies

$$\overline{\partial}_X \bar{f}^{0,0} = D\bar{f}^{1,-1} \quad \text{where} \quad \bar{f}^{1,-1} = \left(c\frac{(\lambda\overline{\lambda})(rw) - (\overline{\lambda}\omega)(\lambda r)}{(\lambda\overline{\lambda})^2} \right). \tag{5.12}$$

($\overline{\partial}_X f^{0,0} = 0$ in this case.) Choosing an operator $\hat{f}^{n+1,-1}$ satisfying $D\hat{f}^{n+1,-1} = -g^{n+1,0}$, the sum $\hat{f}^{n,0} + \hat{f}^{n+1,-1}$ solves the master equation (5.8) to the second line.

Proceeding in a similar manner, one can iteratively determine $\hat{f}^{n+k,-k}$ $(k > 1)$ as follows

$$\hat{g}^{n+k+1,-k} \equiv \overline{\partial}_X \hat{f}^{n+k,-k} \;\rightarrow\; D\hat{g}^{n+k+1,-k} = 0 \;\rightarrow\; \hat{g}^{n+k+1,-k} = -D\hat{f}^{n+k+1,-k-1}\,. \tag{5.13}$$

In general, $D^2 = \{D\,, \overline{\partial}_X\} = 0$ implies that $\hat{g}^{n+k+1,-k}$ defined by the first equation is D-closed. Then since D has no cohomologies at negative degrees, $\hat{g}^{n+k+1,-k}$ is found to be D-exact.

5.2.2 Various descriptions of the b ghost

Before explaining the general relation between BRST and Dolbeault descriptions embedded in the non-minimal BRST cohomology, let us study how the ghost b is described in various cohomologies. Since the quantum BRST cohomology $H^k(D)$ is non-vanishing only at ghost numbers 0 and 1, clearly the ghost b (which is the lowest mass operator in $H^1(D)$) plays a special role among others.

Dolbeault description As explained above, $b \in H^1(D)$ is also in the cohomology of the non-minimal BRST operator $\overline{D} = D + \overline{\partial}_X$. But since inverse powers of $\lambda\overline{\lambda}$ can be used in the non-minimal formulation, operators can have drastically different expressions in this cohomology. Indeed, using the relation

$$b = D\left(\frac{\overline{\lambda}\omega}{2\lambda\overline{\lambda}}\right), \tag{5.14}$$

one can represent b in a gauge where all BRST ghosts are absent:

$$\begin{aligned} b &\simeq -\overline{\partial}_X\left(\frac{\overline{\lambda}\omega}{2\lambda\overline{\lambda}}\right) \\ &= \frac{(\lambda r)(\overline{\lambda}\omega) - (\lambda\overline{\lambda})(r\omega)}{2(\lambda\overline{\lambda})^2}\,. \end{aligned} \tag{5.15}$$

Since there are no ghosts in the final expression, it is easy to identify the corresponding operator in the Dolbeault cohomology:

$$\bar{b} = \frac{(\lambda r)(\overline{\lambda}\omega) - (\lambda\overline{\lambda})(r\omega)}{2(\lambda\overline{\lambda})^2}\,. \tag{5.16}$$

While $\bar{b}$ is trivially $\overline{\partial}_X$-closed (as it is formally a $\overline{\partial}_X$ of a gauge *non*-invariant quantity), it is not a $\overline{\partial}_X$ of a gauge invariant operator and hence is in the Dolbeault cohomology.

Although b and $\bar{b}$ look identical, we emphasize that they are conceptually quite different. In particular, in the space where $\bar{b}$ is defined, the constraint $(\lambda\lambda) = 0$ and the associated gauge invariance are in effect, while they are not for the space where b is defined.

Quantum mechanically, depending on the normal ordering prescription used to define $\bar{b}$, there can be quantum improvement terms of the form $(\lambda\overline{\lambda})^{-2}(\overline{\lambda}\partial r - r\partial\overline{\lambda})$ to assure that $\bar{b}$ is $\overline{\partial}_X$-closed.

Čech description As usual, the Čech and Dolbeault cohomologies are related by the partition of unity on the target space X [8]. As described in appendix A, X can be covered using $2N$ patches U_A ($A = 1 \sim 2N$), where in U_A a certain component of λ which we denote λ^A is non-vanishing. The partition of unity and an associated differential is given by

$$\begin{aligned} \rho_A &= \frac{\overline{\lambda}_A\lambda^A}{\lambda\overline{\lambda}}\,, \qquad\qquad \sum_A \rho_A = 1 \\ \overline{\partial}\rho_A &= \frac{(\lambda\overline{\lambda})r_A\lambda^A - (\lambda r)\overline{\lambda}_A\lambda^A}{(\lambda\overline{\lambda})^2}\,. \end{aligned} \tag{5.17}$$

(Here and hereafter, we do not use the Einstein summation convention for the index A.) Now, the state $\bar{b}$ (5.16) is written as

$$\bar{b} = -\sum_{A,B} \frac{\lambda^{[A}\omega^{B]}}{\lambda^A\lambda^B}\rho_A\overline{\partial}\rho_B\,, \tag{5.18}$$

and hence it corresponds to a Čech 1-cochain

$$\check{b} = (b^{AB}) = -\frac{2\lambda^{[A}\omega^{B]}}{\lambda^A\lambda^B}\,. \tag{5.19}$$

While $\check{b}$ trivially satisfies the cocycle condition as it is formally a $\check{\delta}$ (difference) of two gauge *non*-invariant 0-cochains,

$$\check{b} = \check{\delta}\left(\frac{\omega^A}{2\lambda^A}\right) = \frac{\omega^A}{2\lambda^A} - \frac{\omega^B}{2\lambda^B}\,, \tag{5.20}$$

it is not a difference of gauge invariant 0-cochains and hence is in the Čech cohomology. Of course, this corresponds to the fact that $\bar{b}$ is a $\overline{\partial}_X$ of gauge *non*-invariant function but not a $\overline{\partial}_X$ of gauge invariant function. Using the local coordinates on the overlaps $U_A \cap U_B$, it can be written as

$$\check{b} = (b^{AB}) = (b^{1,\tilde{1}}, b^{1,2}, \cdots)\,, \tag{5.21}$$

$$\text{where} \quad b^{1,\tilde{1}} = \frac{\varrho - \frac{1}{2}(u\cdot u)(u\cdot\partial u)}{g^2(u\cdot u)} = \frac{\tilde{\varrho} - \frac{1}{2}(\tilde{u}\cdot\tilde{u})(\tilde{u}\cdot\partial\tilde{u})}{\tilde{g}^2(\tilde{u}\cdot\tilde{u})}\,, \quad b^{1,2} = \cdots\,. \tag{5.22}$$

5.2.3 Classical mapping between BRST and Dolbeault descriptions and quantum discrepancy

It is straightforward to extend the mapping for the b ghost above to other operators in the cohomology. For the operators in $H^0(D)$ (those corresponding to usual gauge invariant polynomials), the mapping in essence is simply a matter of dropping and recovering the ghost contribution in the t-charge current:[7]

$$J = -\omega\lambda - 2bc \quad \leftrightarrow \quad J = -\omega\lambda\,. \tag{5.23}$$

As for the operators in $H^1(D)$, the mapping works just as in the case of b ghost. One simply gets rid of the b (or its derivative) by using the relation (5.14); this leads to the expression of the non-minimal cohomology element in a gauge where the b ghost is absent (apart from those contained in J's).

Classically, the higher cohomologies $H^k(D)$ $(k > 1)$ are not empty as opposed to the quantum case. For example, a pair of operators $b\partial b$ and $b(\omega\omega)$ with charges q^3t^{-4} are both in the classical cohomology. (Quantum mechanically, those form a BRST doublet.) Using the fact that $\partial^n b = D\partial^n(\overline{\lambda}\omega/2\lambda\overline{\lambda})$ and $\{\overline{\partial}_X, D\} = 0$, however, one can map those higher cohomology elements into the non-minimal gauge by eliminating one unit of ghost charge at a time.

Quantum mechanically, a pair $(\hat{f}, \hat{g})$ of the elements of classical $(D + \overline{\partial}_X)$-cohomology may drop out from the cohomology by forming a doublet $D + \overline{\partial}_X : \hat{f} \to \hat{g}$. Since the curved $\beta\gamma$ and BRST descriptions use different normal ordering prescriptions, it is not assured that this happens if and only if the corresponding elements in the Dolbeault cohomology form a doublet as $\overline{\partial}_X : f \to g$. Indeed, there are mismatches between the two descriptions as explained at the end of section 3.2.

5.2.4 Examples of the mapping

Now, let us illustrate the mapping by translating some specific operators from BRST to Čech-Dolbeault languages.

Example: t-charge current J. First, consider the t-charge current $J = -\omega_i\lambda^i - 2bc$. From

$$\begin{aligned} D\left(\frac{(\overline{\lambda}\omega)c}{\lambda\overline{\lambda}}\right) &= 2bc + \frac{(\overline{\lambda}\omega)(\lambda\lambda)}{\lambda\overline{\lambda}} + \frac{2(\partial\lambda\overline{\lambda})}{\lambda\overline{\lambda}}\,, \\ \overline{\partial}_X\left(\frac{(\overline{\lambda}\omega)c}{\lambda\overline{\lambda}}\right) &= \frac{(r\omega)c}{\lambda\overline{\lambda}} - \frac{(\lambda r)(\overline{\lambda}\omega)c}{(\lambda\overline{\lambda})^2}\,, \end{aligned} \tag{5.24}$$

one finds the following representation of J in the non-minimal BRST cohomology:

$$J \simeq -\omega\lambda + \frac{2(\partial\lambda\overline{\lambda})}{\lambda\overline{\lambda}} + \frac{(\overline{\lambda}\omega)(\lambda\lambda)}{\lambda\overline{\lambda}} + c\frac{(\lambda\overline{\lambda})(r\omega) - (\lambda r)(\overline{\lambda}\omega)}{(\lambda\overline{\lambda})^2}\,. \tag{5.25}$$

[7] Note that BRST ghosts are rotation singlet in our models, so the current N_{ij} does not contain the ghosts.

Apart from the second term, which is a quantum correction, the expression of J is precisely the one we obtained in (5.12) by embedding the Dolbeault cohomology to the non-minimal BRST cohomology.

The normal ordering in (5.25) is that of the free fields. Since $c = (\lambda\lambda) = 0$ in the Dolbeault language, the t-charge current should look like

$$\bar{J} = -\omega\lambda + \frac{2(\partial\lambda\overline{\lambda})}{\lambda\overline{\lambda}}\,, \tag{5.26}$$

where ω, λ and $\overline{\lambda}$ are parameterized by some independent variables. The second term represents some quantum correction, but as ω and λ are no longer free, there seems to be no reason to believe the value of its coefficient. We, however, observe that this value can be understood intuitively as the anomaly coming from the constraint per se.

Note that in a local coordinate one classically has

$$\bar{J} = -\varrho \sim \omega\lambda\,, \tag{5.27}$$

where ϱ is the conjugate to φ parameterizing the length of λ (see appendix A). Quantum mechanically, *if ω and λ were free fields*, this is modified to

$$\bar{J} = -\varrho - \frac{N}{2}\partial\varphi \quad \left(\rightarrow \quad \bar{J}(z)\bar{J}(w) = \frac{-N}{(z-w)^2}\right), \tag{5.28}$$

receiving the correction from the usual free field chiral anomalies. However, some units of the background charge are absent due to constraint, and this is exactly represented by $2(\partial\lambda\overline{\lambda})/(\lambda\overline{\lambda})$. Recalling $(\partial\lambda\overline{\lambda})/(\lambda\overline{\lambda}) \simeq \partial\log(\lambda\overline{\lambda}) \simeq \partial\varphi$, one finally obtains the form of $\bar{J}$, that coincides with the one obtained from the consistent gluing condition:

$$\bar{J} = -\varrho - \frac{N-4}{2}\partial\varphi\,. \tag{5.29}$$

Example: $b\omega_i$. In the minimal BRST description $b\omega_i$ is BRST closed. In fact, it is not difficult to check that it is in the cohomology of D. Now, just like b, $b\omega_i$ is also a representative of a $\overline{D}$-cohomology. But using the formulas

$$\begin{aligned} D\left(\frac{(\overline{\lambda}\omega)\omega_i}{2(\lambda\overline{\lambda})}\right) &= b\omega_i + \frac{b\lambda_i(\overline{\lambda}\omega)}{(\lambda\overline{\lambda})} + \frac{\partial b\overline{\lambda}_i}{(\lambda\overline{\lambda})}\,, \\ D\left(\frac{\lambda_i(\overline{\lambda}\omega)^2}{4(\lambda\overline{\lambda})^2}\right) &= \frac{b\lambda_i(\overline{\lambda}\omega)}{(\lambda\overline{\lambda})} + \frac{\partial b\overline{\lambda}_i}{(\lambda\overline{\lambda})}\,, \end{aligned} \tag{5.30}$$

another representation of $b\omega_i$ (as an element of $(D + \overline{\partial}_X)$-cohomology) can be obtained in which the ghosts are absent:

$$\begin{aligned} b\omega_i &\simeq -\overline{\partial}_X\left(\frac{(\overline{\lambda}\omega)\omega_i}{2(\lambda\overline{\lambda})} - \frac{\lambda_i(\overline{\lambda}\omega)^2}{4(\lambda\overline{\lambda})^2}\right) \\ &= \frac{(\lambda r)(\overline{\lambda}\omega)\omega_i - (\lambda\overline{\lambda})(r\omega)\omega_i}{2(\lambda\overline{\lambda})^2} - \frac{(\lambda r)(\overline{\lambda}\omega)^2\lambda_i - (\lambda\overline{\lambda})(r\omega)(\overline{\lambda}\omega)\lambda_i}{2(\lambda\overline{\lambda})^3}\,. \end{aligned} \tag{5.31}$$

Since $\overline{\partial}_X$ and D commute, the right hand side is necessarily gauge invariant (or D-closed). Also, it is $\overline{\partial}_X$-closed being a $\overline{\partial}_X$ of a gauge *non*-invariant (D-non-closed) operator, but it cannot be written as a $\overline{\partial}_X$ of a gauge invariant operator. Those implies that one can read-off the corresponding element of the Dolbeault cohomology from (5.31). That is, with (ω, λ) being understood as constrained variables,

$$\psi_i = \frac{(\lambda r)(\overline{\lambda}\omega)\omega_i - (\lambda\overline{\lambda})(r\omega)\omega_i}{2(\lambda\overline{\lambda})^2} - \frac{(\lambda r)(\overline{\lambda}\omega)^2\lambda_i - (\lambda\overline{\lambda})(r\omega)(\overline{\lambda}\omega)\lambda_i}{2(\lambda\overline{\lambda})^3} \tag{5.32}$$

is $\overline{\partial}_X$-closed provided the quantum corrections are defined appropriately. But it is not $\overline{\partial}_X$-exact and hence is in the Dolbeault cohomology.

In the Čech language. the corresponding element can be found to be the 1-cochain

$$(\psi_i^{AB}) = \frac{-2\lambda^{[A}\omega^{B]}\omega_i}{\lambda^A\lambda^B} + \frac{2\omega^{(A}\omega^{B)}\lambda_i}{\lambda^A\lambda^B}\,. \tag{5.33}$$

The argument for it being in the Čech cohomology is the same as the Dolbeault case. It satisfies the cocycle condition on the triple overlaps $U_A \cap U_B \cap U_C$,

$$(\psi_i^{AB} - \psi_i^{AC} + \psi_i^{BC}) = 0\,, \tag{5.34}$$

but it is not a coboundary of any gauge invariant operators, and hence is in the Čech cohomology.

6. Summary and discussion

In this paper, we have studied the Hilbert space of the conformal field theories with a simple quadratic constraint $\lambda^i\lambda^i$ $(i = 1 \sim N)$ using both curved $\beta\gamma$ (Čech/Dolbeault) and BRST frameworks. Although there are slight mismatches between the two descriptions due to the quantum ordering problem, we found that their partition functions $\mathrm{Tr}[(-1)^F\cdots]$ agree for $N \geq 4$ models. Since our partition functions in both descriptions are insensitive to quantum corrections, the agreement of the partition functions can be explained by classically relating the elements of the cohomologies of the two formalisms. We showed the classical equivalence of the two cohomologies by embedding them into a combined bigraded cohomology.

Regarding the structure of the Hilbert space itself, we found that the quantum BRST cohomology is non-vanishing only at ghost numbers 0 and 1, and that there is a one-to-one mapping between the two sectors. In terms of the partition function, the mapping between ghost numbers 0 and 1 are summarized as the $*$-conjugation symmetry. We explicitly constructed a non-degenerate inner product that couples the two sectors.

In the BRST language, the lowest mass state in the ghost number 1 cohomology is accounted for by the ghost b itself in the BRST operator $D = \int b(\lambda\lambda)$. In Dolbeault language it corresponds to a 1-form on the constrained surface, and in Čech language it corresponds to a 1-cocycle defined only on the double overlaps of the coordinate charts.

There, however, are several points in the present work that require further clarifications. One of them is to understand the discrepancy between the extrinsic (BRST) and intrinsic (curved $\beta\gamma$) descriptions more precisely.

For the class of models we studied (models on a cone over a base B with the origin removed), we encountered two sources for the discrepancy. Firstly, for lower dimensional models ($N \leq 3$), one finds operators that are globally defined but nevertheless cannot be written as a gauge gauge invariant polynomials in the extrinsic coordinates (ω, λ). We presented an argument for the absence of such operators when the base B has dimensions greater than 1 (i.e. $N \geq 4$), but it would be nice to understand the precise criterion.

At the quantum level, second source for the discrepancy between the BRST and curved $\beta\gamma$ descriptions arises from the different normal ordering prescriptions used in the two. A pair of the elements of the classical BRST cohomology can drop out from the quantum cohomology by forming a BRST doublet, $\hat{g} = D\hat{f}$. In the curved $\beta\gamma$ framework, similar phenomenon occurs when the quantum effect spoils the gluing property of a classical cohomology f. In this case, the failure of gluing is represented by a higher cochain $g = \check{\delta} f$ which is also in the classical cohomology. Since the two frameworks use different normal ordering prescriptions, there are discrepancies between the two phenomena. It would be useful to study if this type of discrepancy can be remedied, for example, by appropriately bosonizing the BRST ghosts.

Another clarification that should be attempted is to explore the one-loop path integral expression for our partition functions. When properly understood, it should be useful for unconvering the origin of the field-antifield and $*$-conjugation symmetries.

Leaving the clarifications of those subtleties to a future work, we list some directions for the extensions of the results obtained in the present paper.

In an accompanying paper [5], we extend the result to the more interesting case of pure spinors. Despite the fact that the pure spinor constraint is infinitely reducible, it will be argued that the structures above carry over almost literally. The only difference is that the ghost numbers at which the cohomology become non-trivial are 0 and 3, instead of 0 and 1. The lowest mass state in the ghost number 3 cohomology carries weight 2 and represents an important term in the reparameterization b-ghost.

Knowing that there can be no cohomologies with ghost numbers greater than 3 is nice for the pure spinor multiloop amplitudes, because it implies that one need not worry about the poles coming from the fusion of many reparameterization b-ghosts. The troublesome poles are necessarily carrying ghost numbers greater than 3 and, modulo the subtleties coming from the divergences at the boundary of moduli spaces, they can be ignored without having have to use the regularization introduced in [3]. It would be interesting to work out how it is actually realized, and the present models might be useful to clarify some aspects of this issue.

Finally, it should be possible to extend our results to the curved $\beta\gamma$ systems with cubic or higher homogeneous constraints (or intersections thereof). For the case of single homogeneous constraint of order L, the result is almost obvious. The $*$-conjugation symmetry relates the states with $q^m t^n g^k$ to those with $q^{m+n+\frac{L(L-1)}{2}} t^{-n-L} g^{L-k-1}$ and it is not difficult to construct the inner product that couples $H^k(D)$ and $H^{L-k-1}(D)$. Note that all cohomologies $H^k(D)$'s with $0 \leq k \leq L-1$ are non-empty having $b\partial b \cdots \partial^{k-1} b$ as the lowest mass element.

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A. Curved $\beta\gamma$ system on the cone $\lambda^i \lambda^i = 0$

In this appendix, we collect some useful formulas for the study of the curved $\beta\gamma$ system on the $N-1$ (complex) dimensional cone

$$X = \{\lambda^i \mid \lambda^i \gamma_{ij} \lambda^j = 0\,, \quad \lambda \neq 0\} \subset \mathbb{C}^N\,, \quad (i,j = 1 \sim N)\,. \tag{A.1}$$

Here, γ_{ij} is a constant symmetric "metric". Below, we diagonalize γ_{ij} and do not distinguish upper and lower indices. Also, we always assume that the origin $\lambda = 0$ is removed so X is a $\mathbb{C}^*$-bundle over a base B.

A.1 Geometry of the cone $\lambda^i \lambda^i = 0$

A.1.1 An open covering

Let us denote

$$\lambda^I = \lambda^i + i\lambda^{i+1}\,, \quad \lambda^{\tilde{I}} = \lambda^i - i\lambda^{i+1}\,. \tag{A.2}$$

Here, i runs over $1 \sim N$ and is defined modulo N. We also use the index A to denote both I and $\tilde{I}$ and use notations

$$\lambda^A = (\lambda^I, \lambda^{\tilde{I}})\,, \quad \lambda_A = \frac{1}{2}(\lambda^{\tilde{I}}, \lambda^I)\,, \quad \sum \lambda^A \lambda_A = \sum \lambda^I \lambda^{\tilde{I}} = \lambda^i \lambda^i\,. \tag{A.3}$$

The cone X can be covered by $2N$ patches $\{U_A\}_{A=1\sim 2N}$, where on a patch at least one of λ^A is non-vanishing:

$$U_A = \{\lambda \mid \lambda^A \neq 0\} \quad \leftrightarrow \quad U_I = \{\lambda \mid \lambda^I \neq 0\} \text{ or } \tilde{U}_{\tilde{I}} = \{\lambda \mid \lambda^{\tilde{I}} \neq 0\}\,. \tag{A.4}$$

On a patch, λ can be parameterized using $N-1$ independent variables (g, u^a), where g parameterizes the overall scale of λ, and u^a's are $N-2$ "angular" variables. For example, on U_1 and $\tilde{U}_{\tilde{1}}$, the local coordinates are $(g_{(1)}, u^a_{(1)})_{a=3\sim N}$ and $(\tilde{g}_{(1)}, \tilde{u}^a_{(\tilde{1})})_{a=3\sim N}$ respectively, and λ is parameterized as (omitting the subscript (1) and $(\tilde{1})$ for simplicity)

$$\begin{aligned} U_1 &: (\lambda^1 + i\lambda^2, \lambda^1 - i\lambda^2, \lambda^a) = (g, g(u\cdot u), igu^a)\,, \\ \tilde{U}_1 &: (\lambda^1 + i\lambda^2, \lambda^1 - i\lambda^2, \lambda^a) = (\tilde{g}(\tilde{u}\cdot\tilde{u}), \tilde{g}, i\tilde{g}\tilde{u}^a)\,. \end{aligned} \tag{A.5}$$

Variables (g, u^a) on other patches are defined in a similar manner.

A.1.2 Coordinate transformation

The transformations among the coordinates above are readily computed. We here give the transition functions on $U_1 \cap \tilde{U}_{\tilde{1}}$, $U_1 \cap U_2$ and $U_{\tilde{1}} \cap U_2$.

On the overlap $U_1 \cap \tilde{U}_1$, both $\lambda^1 + i\lambda^2 = g = \tilde{g}(\tilde{u} \cdot \tilde{u})$ and $\lambda^1 - i\lambda^2 = \tilde{g} = g(u \cdot u)$ are non-vanishing. Hence, $(u \cdot u)$ and $(\tilde{u} \cdot \tilde{u})$ are also non-vanishing and the two coordinates are related by

$$(g, u^a) = (\tilde{g}(\tilde{u} \cdot \tilde{u}), \tilde{u}^a(\tilde{u} \cdot \tilde{u})^{-1}) , \quad (\tilde{g}, \tilde{u}^a) = (g(u \cdot u), u^a(u \cdot u)^{-1}) . \tag{A.6}$$

To describe the transformation on the overlap between U_1 and U_2, let us temporarily denote

$$(G, U^a) = (g_{(2)}, u^a_{(2)}) , \quad (a = 4 \sim (N+1) = 1, 4 \sim N) . \tag{A.7}$$

On the overlap $U_1 \cap U_2$, g and G as well as $(1 - 2iu_3 - u \cdot u)$ and $(1 + 2U_1 + U \cdot U)$ are non-vanishing and (g, u^a) and (G, U^a) are related as

$$\begin{aligned} g &= \frac{i}{2} G(1 + 2U_1 + U \cdot U) , & u_3 &= \frac{i(1 - U \cdot U)}{1 + 2U_1 + U \cdot U} , & u_a &= \frac{2iU_a}{1 + 2U_1 + U \cdot U} , \\ G &= -\frac{i}{2} g(1 - 2iu_3 - u \cdot u) , & U_1 &= \frac{1 + u \cdot u}{1 - 2iu_3 - u \cdot u} , & U_a &= \frac{2iu_a}{1 - 2iu_3 - u \cdot u} . \end{aligned} \tag{A.8}$$

Similarly, the relation between $(\tilde{g}, \tilde{u})$ and (G, U) on the overlap $\tilde{U}_{\tilde{1}} \cap U_2$ are given by

$$\begin{aligned} \tilde{g} &= -\frac{i}{2} G(1 - 2U_1 + U \cdot U) , & \tilde{u}_3 &= \frac{-i(1 - U \cdot U)}{1 - 2U_1 + U \cdot U} , & \tilde{u}_a &= \frac{2iU_a}{1 - 2U_1 + U \cdot U} , \\ G &= \frac{i}{2} \tilde{g}(1 + 2i\tilde{u}_3 - \tilde{u} \cdot \tilde{u}) , & U_1 &= \frac{-1 - \tilde{u} \cdot \tilde{u}}{1 + 2i\tilde{u}_3 - \tilde{u} \cdot \tilde{u}} , & U_a &= \frac{-2i\tilde{u} \cdot \tilde{u}}{1 + 2i\tilde{u}_3 - \tilde{u} \cdot \tilde{u}} . \end{aligned} \tag{A.9}$$

One can easily check the consistency of the transformations on the triple overlap $U_1 \cap U_{\tilde{1}} \cap U_2$.

A.1.3 Partition of unity

By introducing the non-minimal variables $\overline{\lambda}_i$ (complex conjugates to λ^i), a partition of unity on X can be constructed explicitly as

$$\rho_A = \frac{\lambda^A \overline{\lambda}_A}{\lambda \overline{\lambda}} , \quad \left(\lambda \overline{\lambda} = \lambda^i \overline{\lambda}_i = \sum_A \lambda^A \overline{\lambda}_A = \sum_A g_{(A)} \overline{g}_{(A)} \right) . \tag{A.10}$$

Clearly, $\{\rho_A\}$ is subordinate to the covering $\{U_A\}$, that is, $\rho_A = 0$ outside the patch U_A. The derivative of ρ_A is

$$\overline{\partial} \rho_A = \frac{(\lambda \overline{\lambda}) r_A \lambda^A - (\lambda r) \overline{\lambda}_A \lambda^A}{(\lambda \overline{\lambda})^2} . \tag{A.11}$$

A Čech n-cochain $(f^{A_0 A_1 \cdots A_n})$ and the corresponding n-form in Dolbeault language $\bar{f}$ are related as

$$\bar{f} = \frac{1}{(n+1)!} \sum f^{A_0 A_1 \cdots A_n} \rho_{A_0} \mathrm{d}\rho_{A_1} \cdots \mathrm{d}\rho_{A_n} . \tag{A.12}$$

A.2 $\beta\gamma$ system on the cone $\lambda^i\lambda^i = 0$

A.2.1 Free curved $\beta\gamma$ system on a patch

On a patch U_A, the conjugates to (g, u^a) are denoted as (h, v_a) and they satisfy the free field operator product expansions

$$h(z)g(w) = \frac{-1}{z-w}\,, \quad v_a(z)u^b(w) = \frac{-\delta_a{}^b}{z-w}\,. \tag{A.13}$$

Since g is non-vanishing, one can instead use $\varphi = \log g$ and its conjugate ϱ satisfying

$$\varrho(z)\varphi(w) = \frac{-1}{z-w}\,. \tag{A.14}$$

A.2.2 Transformation of momenta

On an overlap $U_A \cap U_B$, the momenta on U_A and those on U_B are related as

$$\vec{v}_{(B)} = :\vec{v}_{(A)}(\tau_{AB}): + (\phi_{AB})\partial\vec{u}_{(A)}\,, \tag{A.15}$$

where we denoted $\vec{u}_{(A)} = (\varphi_{(A)}, u_{(A)})$ and $\vec{v}_{(A)} = (\varrho_{(A)}, v_{(A)})$ for simplicity. (τ_{AB}) is the Jacobian $(\partial\vec{u}_A/\partial\vec{u}_B)$, and the matrix (ϕ_{AB}) is defined so that $\vec{v}_{(B)}$'s do not have singular operator products among themselves.

On the overlap $U_1 \cap \tilde{U}_{\tilde{1}}$, the momenta are related as

$$\begin{aligned}\tilde{\varrho} &= \varrho - \frac{(N-4)}{2}\partial\log(u\cdot u)\,,\\ \tilde{v}_a &= 2\varrho u_a + (u\cdot u)v_a - 2(u\cdot v)u_a + 4\partial u_a - (N-4)(\partial\varphi)u_a\,.\end{aligned} \tag{A.16}$$

Since $\tilde{v}_a$ generates a translation on $\tilde{U}_{\tilde{1}}$, it should agree with the corresponding rotation generator N_a^- in the coordinate U_1. This indeed is the case (see below).

On the overlap $U_1 \cap U_2$, the momenta in U_2 which we denote $(R, V_1, V_{a'})_{a'=4\sim N}$ are

$$\begin{aligned}R &= \varrho - \frac{(N-4)}{4}\partial\log(1-2iu_3-u_3^2+u_{c'}u_{c'})\,,\\ V_1 &= (1-iu_3)(\varrho - v_{c'}u_{c'}) - \frac{i}{2}(1-2iu_3-u_3^2+u_{c'}u_{c'})v_3\\ &\qquad\qquad - 2i\partial u_3 - \frac{(N-4)}{2}(1-iu_3)\partial\varphi\\ &= -N + \frac{i}{2}N_3^+ + \frac{i}{2}N_3^-\,,\\ V_{a'} &= i\varrho u_{a'} + (v_3 - iv_3u_3 - iv_{c'}u_{c'})u_{a'} - \frac{i}{2}(1-2iu_3-u_3^2-u_{c'}v_{c'})v_{a'}\\ &\qquad\qquad + 2i\partial u_{a'} - \frac{(N-4)i}{2}u_{a'}\partial\varphi\\ &= \frac{i}{2}N_{a'}^+ - N_{3a'} - \frac{i}{2}N_{a'}^-\,.\end{aligned} \tag{A.17}$$

Again, $V_{1,a'}$ corresponds to certain linear combinations of the rotation currents.

The quantum correction part $(\phi_{AB})\partial\vec{u}_A$ in (A.15) cannot be defined consistently to satisfy the cocycle condition $(\phi_{AC})(\phi_{BC})(\phi_{AB}) = 1$, unless a closed 2-form valued 2-cocycle

$$(\psi_{ABC}) = \mathrm{tr}(\tau_{AB} \wedge \mathrm{d}\tau_{BC} \wedge \mathrm{d}\tau_{CA}) \tag{A.18}$$

represents a trivial class in the Čech cohomology. On the triple overlap $U_1 \cap \tilde{U}_{\tilde{1}} \cap U_2$, ψ is given by

$$\begin{aligned}(\psi_{1\tilde{1}2}) &= \mathrm{tr}(\tau_{1\tilde{1}} \wedge \mathrm{d}\tau_{\tilde{1}2} \wedge \mathrm{d}\tau_{21}) \\ &= \sum_{a'=4}^{N} \frac{4i(N-4)u_{a'}\mathrm{d}u_3 \wedge \mathrm{d}u_{a'}}{(u\cdot u)(1-2iu_3+u\cdot u)} \\ &= (N-4)\mathrm{d}\log(u\cdot u) \wedge \mathrm{d}\log(1-2iu_3+u\cdot u)\,.\end{aligned} \tag{A.19}$$

This expression of ψ tells us two things. First, note that the right hand side only includes the coordinates of the base B. This is a general feature of the models with a $\mathbb{C}^*$-fiber and $\psi_{1\tilde{1}2}$ coincides with the obstruction for the model on the base B. On B, there is no way to rewrite (A.19) as a coboundary of a 2-cochain holomorphic in $U_A \cap U_B$ (restricted to B), so the curved $\beta\gamma$ system with target space B is anomalous, i.e. the momenta cannot be glued consistently.

At the same time, we find from (A.19) that ψ is in fact trivial on X, as it is a coboundary of 2-cochains holomorphic in $U_1 \cap U_{\tilde{1}}$, $U_1 \cap U_2$ and $U_{\tilde{1}} \cap U_2$:

$$\begin{aligned}\psi_{1\tilde{1}2} &\propto \check{\delta}(\mathrm{d}\varphi \wedge \mathrm{d}\tilde{\varphi},\, \mathrm{d}\varphi \wedge \mathrm{d}\Phi,\, \mathrm{d}\tilde{\varphi} \wedge \mathrm{d}\Phi) \\ &= \mathrm{d}\varphi \wedge \mathrm{d}\log(u\cdot u) - \mathrm{d}\varphi \wedge \mathrm{d}\log(1-2iu_3+u\cdot u) \\ &\qquad + \mathrm{d}(\varphi + \log(u\cdot u)) \wedge \mathrm{d}(\varphi + \log(1-2iu_3+u\cdot u)) \\ &= \mathrm{d}\log(u\cdot u) \wedge \mathrm{d}\log(1-2iu_3+u\cdot u)\,.\end{aligned} \tag{A.20}$$

That is, the obstruction (ψ_{ABC}) represents a trivial class $\check{\delta}(\mathrm{d}\varphi_A \wedge \mathrm{d}\varphi_B)$ in the Čech cohomology, so the momenta on X (unlike those restricted on B) can be glued consistently.

A.2.3 Symmetry currents

The cone X is invariant under the rescaling and rotations of λ. In a given patch, the corresponding currents take the following forms:

$$\begin{aligned} J &= -\varrho - \frac{n-4}{2}\partial\varphi\,, & & \\ N &= (v\cdot u) - J'\,, & N_{ab} &= -v_a u_b + v_b u_a\,, \\ N_a^+ &= -v_a\,, & N_a^- &= 2(v\cdot u)u^a - (u\cdot u)v^a - 2J'u^a - 4\partial u^a\,. \end{aligned} \tag{A.21}$$

Here, $J' = \varrho - \frac{n-4}{2}\partial\varphi$ is defined so that $J(z)J'(w)$ have no poles, and we temporarily denoted the number of λ components by n, to avoid the confusion with the operator N that generates $\mathrm{U}(1) \subset \mathrm{SO}(n)$.

(J, N) form the $\mathrm{U}(1)_t \times \mathrm{SO}(n)$ current algebra with levels $(4-n, -2)$:

$$
\begin{aligned}
J(z)J(w) &= \frac{4-n}{(z-w)^2}\,, \\
N(z)N(w) &= \frac{-2}{(z-w)^2}\,, \\
N(z)N_a^{\pm}(w) &= \frac{\pm N_a^{\pm}(w)}{z-w}\,, \\
N_a^{+}(z)N_b^{-}(w) &= \frac{-4\delta_{ab}}{(z-w)^2} + \frac{2N_{ab}(w) + 2\delta_{ab}N(w)}{z-w}\,, \\
N_a^{+}(z)N_{bc}(w) &= \frac{2\delta_{a[b}N_{c]}^{+}(w)}{z-w}\,, \\
N_a^{-}(z)N_{bc}(w) &= \frac{-2\delta_{a[b}N_{c]}^{-}(w)}{z-w}\,, \\
N_{ab}(z)N_{cd}(w) &= \frac{-2(\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd})}{(z-w)^2} + \frac{2\delta_{a[c}N_{d]b}(w) - 2\delta_{b[c}N_{d]a}(w)}{z-w}\,, \\
(\text{others}) &= \text{regular}\,.
\end{aligned}
\tag{A.22}
$$

Note that the rescaling by J commutes with the rotations by N.

A.2.4 Energy-momentum tensor

Finally, using the coordinate above, one can construct the nowhere vanishing holomorphic top form Ω on X. Choosing the orientation of the coordinates consistently, it takes the form

$$
\Omega = \mathrm{e}^{(N-2)\varphi} \mathrm{d}\varphi \wedge \mathrm{d}u^3 \wedge \cdots \wedge \mathrm{d}u^N\,, \tag{A.23}
$$

in all coordinate patches. Definition of Ω is purely geometric and it is straightforward to check that it transforms covariantly on the overlaps. Hence, X is a (non-compact) Calabi-Yau space and one can define a globally defined conformal field theory for which the energy-momentum tensor is given by gluing

$$
\begin{aligned}
T &= T_{\text{naive}} - \frac{1}{2}\partial^2 \log(\mathrm{e}^{(N-2)\varphi}) \\
&= -\varrho\partial\varphi - v_a \partial u^a - \frac{(N-2)}{2}\partial^2\varphi\,.
\end{aligned}
\tag{A.24}
$$

Note that the background charge for φ obtained here is consistent with the t-charge anomaly

$$
J(z)T(w) = \frac{2-N}{(z-w)^3} + \frac{J(w)}{(z-w)^2}\,. \tag{A.25}
$$

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